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A Survey on The Vector Lyapunov Functions And Practical Stability Of Nonlinear Impulsive Caputo Fractional Differential Equations Via New Modelled Generalized Dini Derivative

Atsu, Jeremiah Ugeh
atsujeremiah@gmail.com
Department of Mathematics,
University of Cross River State, Calabar, Nigeria.

Ante, Jackson Efiong
jackson.ante@topfaith.edu.ng
Department of Mathematics,
Topfaith University, Mkpatak, Nigeria.

Inyang, Augustine Brendan
ab.inyang@topfaith.edu.ng
Department of Business Administration,
Topfaith University, Mkpatak, Nigeria.

Akpan, Ubong Dominic
akpanubong2210@yahoo.com
Department of Mathematics,
Akwa Ibom State University, Ikot Akpaden, Nigeria.

ABSTRACT
This paper examines the practical stability of the trivial solution of a nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse using a class of piecewise continuous Lyapunov functions which generalizes the vector Lyapunov functions. Together with comparison results, sufficient conditions for the practical stability of the impulsive Caputo fractional order systems are established. Results obtained extends and improves on existing results.

Keywords: *practical stability, Caputo derivative, impulse, vector Lyapunov functions, fractional differential equations.*
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I. INTRODUCTION

Fractional calculus has been 300years old history, and its development is mainly focused in the pure mathematical field [27]. The earliest – more or less systematic studies of the concept, seem to have been made in the 19th century by Liouville, Riemann, Leibnitz, Caputo, etc. [37], and in the last four decades, it was observed that the advancement of fractional calculus has enabled the description of complex system behaviors through fractional differential systems (including many physical phenomenon having memory and genetic characteristics), thus providing new insights into their dynamics.

One of the trends in the stability theory of solutions of differential equations is the so-called practical stability [8, 27, 31, 32, 33]. This aspect of stability was introduced by [24] and it is used in estimating the worst-case transient and steady-state responses together with verifying point-wise in time constraints imposed on the solution path or the trajectory curve.. Fundamental results have been obtained for fractional order derivative using the auxiliary Lyapunov's functions which are analogues of vector Lyapunov functions, by means of the comparison method.

Alongside the development of the theory of practical stability in recent years is the mathematical theory of impulsive differential equations which have experienced a massive research attention and development. Now, the theory of impulsive differential equations is richer than the corresponding theory of differential equations

[16] as they constitute very important models for describing the true state of several real life processes and phenomena since many evolution processes are characterized by the fact that, at certain moments of time, they experience a change of state abruptly. These processes are assumed to be subject to short term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. For instance, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems do exhibit impulsive effects [16].

Moreover, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [35], and one of the most versatile methods in the study of the stability properties of impulsive systems is the Lyapunov function (Lyapunov second method) The method was originally developed for studying the stability of a fixed point of an autonomous or nonautonomous differential equations. However, as was argued in [29], it was then extended from fixed points to sets, from differential equation to dynamical systems and to stochastic equations.

Suffice to say that the novelty of the Lyapunov's second method over other methods of examining stability properties of impulsive differential systems like the Razumikhin technique, the use of matrix inequality, etc. stems from the fact that the method allows us to examine the stability of solutions without first solving the given differential equation by seeking an appropriate continuously differentiable function (called Lyapunov function) that is positive definite and whose time derivative along the trajectory curve is negative semidefinite.

The stability of the zero solution of impulsive differential equations have been extensively studied in [11], [30]. Furthermore, the study of stability of fractional order systems is quite recent and one of the main difficulties in the application of the Lyapunov function to fractional order differential equations is the appropriate definition of its derivative among the fractional differential equations. Thus, to allay this problem, [1] adopted the choice of a Lyapunov function that is continuously differential, with a re-definition of the Dini derivative for the given fractional order system. This choice was necessary because, the choice of the Dini derivative for the fractional differential equation as was used in [17], [18] had some “restrictions and difficulties” (see [1] and the references therein).

Now, the stability of fractional order systems systems using the scalar Lyapunov function have been examined in [1], [2], [5], [9], [41]. Using the generalized Caputo fractional Dini derivative and scalar impulsive fractional differential equations, [2] established the comparison results together with sufficient conditions for the stability properties of impulsive fractional differential equations. However, the set-back in this approach arise from the fact that, when the system becomes complex or large, the scalar Lyapunov functions lacks a definite algorithm for handling such systems. Due to this pitfall, the use of vector Lyapunov functions becomes very necessary because of its ability to handle complex systems as well as large scale dynamical systems. The method of vector Lyapunov function involves splitting the Lyapunov functions into several components, so that each of the components can adequately describe the system state. In this way, Lyapunov functions are easily constructed, and the conditions ensuring the required stability are less restrictive (see [40] and the references therein). Fundamental results for the stability of impulsive Caputo fractional differential equations have been examined in [1] and [2].

In this paper, the practical stability of impulsive Caputo fractional order systems is considered, and by means of the comparison principle, sufficient conditions for the practical stability of impulsive fractional order systems is established using a class of piecewise continuous Lyapunov functions. An illustrative example is given to confirm the suitability of the obtained results.

II. Preliminary Notes and Definitions

Fractional calculus is seen as a natural generalization of the classical calculus of integer order and thereby allows for the extension of the traditional concepts of derivative and integral to functions with fractional orders. By this extension, functions with noninteger orders are much more flexible in describing real world systems (see [14], [25], [26], [34], [37]).

There are several definitions of fractional derivatives and fractional integrals.

General case. Let the number $n-1<\beta<n, \beta>0$ be given, where n is a natural number, and $\Gamma(.)$ denotes the Gamma function.

Definition 2.1.

The Riemann Liouville fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$$ {}_{t_0}^{RL}D_t^{\beta}\gamma(t)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma(s)ds, t\geq t_0 $$

Definition 2.2.
The Caputo fractional derivative of order β of $\gamma(t)$ is given by (see [35])

$${}_{t_0}^{C}D_t^{\beta}\gamma(t) = \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-s)^{n-\beta-1}\gamma^{(n)}(s)ds, t \ge t_0$$

The Caputo derivative has many properties that are similar to those of the standard derivatives which make them easier to understand and apply. Also, the initial conditions of the Caputo fractional order derivative are also easier to interpret in physical context.

Definition 2.3.
The Grunwald-Letnikov fractional derivative of order β of $\gamma(t)$ is given by (See [1])

$${}^{GL}D_0^{\beta}\gamma(t) = \lim_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t \ge t_0$$

and

Definition 2.4. The Grunwald-Letnikov fractional Dini derivative of order β of $\gamma(t)$ is given by (See [1])

$${}^{GL}D_0^{\beta}\gamma(t) = \limsup_{h\to 0^+}\frac{1}{h^{\beta}}\sum_{r=0}^{\left[\frac{t-t_0}{h}\right]}(-1)^r\,{}^{\beta}C_r\gamma(t-rh),\ t \ge t_0$$

where ${}^{\beta}C_r$ are the binomial coefficients and $\left[\frac{t-t_0}{h}\right]$ denotes the integer part of $\frac{t-t_0.}{h}$

Particular case (when n=1). In most applications, the order of β is often less than 1, so that $\beta \in (0,1)$. For simplicity of notation, we will use ${}^{C}D^{\beta}$ instead of ${}_{t_0}^{C}D^{\beta}$ and the Caputo fractional derivative of order β of the function $\gamma(t)$ is

$${}^{C}D^{\beta}\gamma(t) = \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{-\beta}\gamma'\,ds,\ \ t \ge t_0$$

(2.1)

III. IMPULSES IN FRACTIONAL DIFFERENTIAL EQUATIONS

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < \beta < 1$.

$$\begin{aligned}{}^{C}D^{\beta}\gamma(t) &= f(t,\gamma),\ t \ge t_0,\\ \gamma(t_0) &= \gamma_0,\end{aligned} \tag{3.1}$$

where $\gamma \in R^N$, $f \in C[R_+ \times R^N, R^N]$, $f(t,0) \equiv 0$ *and* $(t_0, x_0) \in R_+ \times R^N$.

Some sufficient conditions for the existence of the global solutions to (3.1) are considered in [7], [11], [22], [23], [29], [35], [42].

The IVP for FrDE (3.1) is equivalent to the following Volterra integral equation (See [2]),

$$\gamma(t) = \gamma_0 + \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}f(s,\gamma(s))ds,\ t \ge t_0 \tag{3.2}$$

Consider the IVP for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < \beta < 1$,

$$\begin{aligned}{}^{C}D^{\beta}\gamma(t) &= f(t,\gamma),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,...\\ \Delta\gamma &= I_k(\gamma(t_k)),\ t = t_k,\ k \in N,\\ \gamma(t_0^+) &= \gamma_0,\end{aligned}$$

(3.3)

where $\gamma,\gamma_0 \in R^N,\ f \in C[R_+ \times R^N, R^N],\ and\, t_0 \in R_+\ I_k : R^N \to R^N,\ k = 1,2,....$
under the following assumptions:
$(i)\, 0 < t_1 < t_2 < ... < t_k < ...,\ and\ t_k \to \infty\ as\, k \to \infty;$
$(ii)\, f : R_+ \times R^N \to R^N$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $x \in R^N, k = 1,2,...,$ and $\lim_{(t,y)\to(t_k^+,\gamma)} f(t,y) = f(t_k^+,\gamma)$ exists;
$(iii)\, I_k \times R^N \to R^N$
In this paper, we assume that $f(t,0) \equiv 0,\ I_k(0) \equiv 0$ for all k so that we have trivial solution for (3.3), and the points $t_k,\ k = 1,2,...$ are fixed such that $t_1 < t_2 < ....\ and\ \lim_{k\to\infty} t_k = \infty.$ The system (3.3) with initial condition $\gamma(t_0) = \gamma_0$ is assumed to have a solution $\gamma(t;t_0,\gamma_0) \in PC^\beta([t_0,\infty), R^N).$

Remark 3. 1. The second equation in (3.3) is called the impulsive condition, and the function $I_k(\gamma(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 3.1. Let $\Omega : R_+ \times R^N \to R^N$ Then Ω is said to belong to class χ if,
(i) Ω is continuous in $(t_{k-1}, t_k]$ and for each $\gamma \in R^N$ and $\lim_{(t,y)\to(t_k^+,\gamma)} \Omega(t,y) = \Omega(t_k^+,\gamma)$ exists;
(ii) Ω is locally Lipschitz with respect to its second argument x and $\Omega(t,0) \equiv 0$
Now, for any function $\Omega(t,\gamma) \in PC([t_0,\infty) \times \xi, R_+^N).$ we define the Caputo fractional Dini derivative as:

$${}^{C}D_+^\beta \Omega(t,\gamma) = \limsup_{h\to 0^+} \frac{1}{h^\beta}\{\Omega(t,\gamma) - \Omega(t_0,\gamma_0) - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r[\Omega(t - rh, \gamma - h^\beta f(t,\gamma) - \Omega(t_0,\gamma_0)]\} \quad (3.4)$$

$t \ge t_0$ where $t \in [t_0,\infty),\ \gamma,\gamma_0 \in \xi,\ \xi \in R^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T].$

Definition 3.2. A function $g \in PC[R^n, R^n]$ is said to be quasimonotone nondecreasing in γ, if $\gamma \le y$ and $\gamma_i = y_i$ for $1 \le i \le n$ implies $g_i(\gamma) = g_i(y).$

Definition 3.3. The zero solution of (3.3) is said to be:
(*PS1*) practically stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ continuous in t_0 such that for any $\gamma_0 \in R^N,\ \|\gamma_0\| \le \delta$ implies $\gamma_0 \in R^N\ \|\gamma(t,t_0,\gamma_0)\| < \varepsilon$ for $t \ge t_0;$
(*PS2*) uniformly practically stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$, continuous in t_0 such that for any $\gamma_0 \in R^N,\ \|\gamma_0\| \le \delta$ implies $\gamma_0 \in R^N\ \|\gamma(t,t_0,\gamma_0)\| < \varepsilon$ for $t \ge t_0;$
(*PS3*) asymptotically practically stable if it is stable and if for each $\varepsilon > 0\ and\, t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0,\varepsilon)$ such that for $t \ge t_0 + T$ and $\|\gamma_0\| \le \delta$ implies $\|\gamma(t,t_0,\gamma_0)\| < \varepsilon;$
(*PS4*) uniformly asymptotically practically stable if it is uniformly stable and $\delta_0 = \delta_0(\in)$ and $T = T(\varepsilon)$ such that for $t \ge t_0 + T$, the inequality $\|\gamma_0\| \le \delta$ implies $\|\gamma(t,t_0,\gamma_0)\| < \varepsilon.$

Definition 3.4. A function $a(r)$ is said to belong to the class K if $a \in PC([0,\psi), R_+),\ a(0) = 0,$ and $a(r)$ is strictly monotone increasing in $r.$
In this paper, we define the following sets:

$$\overline{S}_\psi = \{\gamma \in R^N : \|\gamma\| \le \psi\}$$
$$S_\psi = \{\gamma \in R^N : \|\gamma\| < \psi\}$$

Suffice to say that the inequalities between vectors are understood to be component-wise inequalities.
We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$
\begin{aligned}
{}_{t_0}^{C}D^{\beta}u &= g(t,u),\ t \ge t_0,\ t \ne t_k,\ k = 1,2,\ldots \\
\Delta u &= \psi_k(u(t_k)),\ t = t_k,\ k \in N, \\
u(t_0^+) &= u_0,
\end{aligned}
\tag{3.5}
$$

existing for $t \ge t_0$, $u \in R^n$, $R_+ = [t_0,\infty)$, $g : R_+ \times R_+^n \to R^n$, $g(t,0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0,u_0) \in R_+ \times R^n$, the system (*3.5*) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t,t_0,u_0) \in PC^{\beta}([t_0,\infty), R^n)$.

Lemma 3.2. Assume $m \in PC([t_0,T] \times \overline{S}_{\psi}, R^N)$. and suppose there exists $t^* \in [t_0,T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*,\alpha_1) = m(t^*,\alpha_2)$ and $m(t,\alpha_1) < m(t,\alpha_2)$ for $t_0 \le t < t^*$. Then if the Caputo fractional Dini derivative of m exists at t^*, then the inequality ${}^{C}D_{+}^{\beta}m(t^*,\alpha_1) - {}^{C}D_{+}^{\beta}m(t^*,\alpha_2) > 0$ holds.

Proof. Let $\Omega(t,\gamma) = m(t,\alpha_1) - m(t,\alpha_2)$. Applying (*3.4*), we have

$$
\begin{aligned}
{}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \{ & [m(t^*,\alpha_1) - m(t^*,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\
& - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [m(t^* - rh,\alpha_1) - m(t^* - rh,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \}
\end{aligned}
$$

when $\alpha_1 = \alpha_2$ we have

$$
\begin{aligned}
{}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \{ & -[m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\
& - \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r+1}\, {}^{\beta}C_r [m(t^* - rh,\alpha_1) - m(t^* - rh,\alpha_2)] - [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \}
\end{aligned}
$$

$$
\begin{aligned}
{}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = & -\limsup_{h\to 0^+} \frac{1}{h^{\beta}} [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\
& + \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t^* - rh,\alpha_1) - m(t^* - rh,\alpha_2)] \\
& - \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t_0,\alpha_1) - m(t_0,\alpha_2)]
\end{aligned}
$$

$$
\begin{aligned}
{}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = & -\limsup_{h\to 0^+} \frac{1}{h^{\beta}} [m(t_0,\alpha_1) - m(t_0,\alpha_2)] \\
& - \limsup_{h\to 0^+} \frac{1}{h^{\beta}} \sum_{r=1}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t_0,\alpha_1) - m(t_0,\alpha_2)]
\end{aligned}
$$

$$
{}^{C}D_{+}^{\beta}(m(t^*,\alpha_1) - m(t^*,\alpha_2)) = -\limsup_{h\to 0^+} \frac{1}{h^{\beta}} \sum_{r=0}^{\left[\frac{t-t_0}{h}\right]} (-1)^{r}\, {}^{\beta}C_r [m(t_0,\alpha_1) - m(t_0,\alpha_2)]
$$

Applying equation (3.8) in [1], we have

$$^{C}D_{+}^{\beta}m(t^{*},\alpha_1) - {}^{C}D_{+}^{\beta}m(t^{*},\alpha_2) = -\frac{(t-t_0)^{-\beta}}{\Gamma(1-\beta)}\left[m(t^{*},\alpha_1) - m(t^{*},\alpha_2)\right]$$

By the lemma, we have that

$$m(t,\alpha_1) - m(t,\alpha_2) < 0 \quad for\ t_0 \le t \le t^{*}$$

And so, it follows that

$$^{C}D_{+}^{\beta}m(t^{*},\alpha_1) - {}^{C}D_{+}^{\beta}m(t^{*},\alpha_2) > 0$$

Note that some existence results for (3.5) are given in [11], [13] and [14].

Remark 3.3. Lemma 3.2 extends Lemma 1 in [1], where the vectors $m(t,\alpha_1)$ *and* $m(t,\alpha_2)$ are compared component-wise.

In the following, we establish the comparison result for the system (3.3).

IV. FRACTIONAL DIFFERENTIAL INEQUALITIES AND COMPARISON RESULTS FOR VECTOR FRACTIONAL DIFFERENTIAL EQUATION

In this section, again we assume that $0 < \beta < 1.$

Theorem 4.1 (Comparison Result).

Assume that

i) $g \in PC[R_+ \times R_+^n, R^n]$ and is continuous in $(t_{k-1}, t_k],\ k = 1,2,...$ and $g(t,u)$ is quasimonotone non-decreasing in u for each $u \in R^n$ and $\lim\limits_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

ii) $\Omega \in PC[R_+ \times R^N, R_+^N]$ and $\Omega \in \chi$ such that

$$^{C}D_{+}^{\beta}\Omega(t,\gamma) \le g(t,\Omega(t,\gamma)),\ \ (t,\gamma) \in R_+ \times R^N$$

and

$\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t,\gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

iii) $\vartheta(t) = \vartheta(t,t_0,u_0) \in PC^{\beta}([t_0,T],R^n)$ be the maximal solution of the IVP for the IFrDE system (3.5)

Then,

$$\Omega(t,\gamma(t)) \le \vartheta(t),\ \ t \ge t_0, \tag{4.1}$$

where $\gamma(t) = \gamma(t,t_0,\gamma_0) \in PC^{\beta}([t_0,T],R^N)$ is any solution of (3.3) existing on $[t_0,\infty),$ provided that

$$\Omega(t_0^+,\gamma_0) \le u_0 \tag{4.2}$$

Proof.

Let $\eta \in S_\psi$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations,

$$^{C}D^{\beta}u = g(t,u) + \eta, \quad for\ t \in [t_0,\infty)$$
$$u(t_0^+) = u_0 + \eta, \tag{4.3}$$

for $t \in [t_0,\infty).$

The function $u_\eta(t,\alpha)$ is a solution of (4.3), where $\alpha > 0$ the fractional differential equation (3.5) if and only if it satisfies the Volterra fractional integral equation,

$$u_\eta(t,\alpha) = u_0 + \eta + \tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,u_\eta(s,\alpha))+\alpha)ds,\ \ t\in[t_0,\infty). \tag{4.4}$$

Let the function $m(t,\alpha)\in PC([t_0,T]\times S_\psi,\ R_+^N)$ be defined as $m(t,\alpha)=\Omega(t,\gamma^*(t))$

We now prove that

$$m(t,\alpha) < u_\eta(t,\alpha)\quad for\quad t\in[t_0,\infty)$$

(4.5)

Observe that the inequality (4.5) holds whenever $t = t_0$ *i.e.*

$$m(t_0,\alpha) = \Omega(t_0,\gamma_0) \le u_0 < u_\eta(t_0,\alpha)$$

Assume that the inequality (4.5) is not true, then there exists a point $t_1 > t_0$ such that

$$m(t_1,\alpha) = u_\eta(t_1,\alpha)\ \ and\ \ m(t,\alpha) < u_\eta(t,\alpha)\ \text{for}\ \ t\in[t_0,t_1).$$

It follows from Lemma 3.2 that

$$^{C}D_{+}^{\beta}(m(t_1,\alpha)-u_\eta(t_1,\alpha)) > 0\ \ i.e.$$

$$^{C}D_{+}^{\beta}(m(t_1,\alpha)) > {}^{C}D_{+}^{\beta}u_\eta(t_1,\alpha)$$

$$^{C}D_{+}^{\beta}\Omega(t_1,\gamma(t_1)) > {}^{C}D_{+}^{\beta}u_\eta(t_1,\alpha)$$

and using (4.3) we arrive at

$$^{C}D_{+}^{\beta}\Omega(t_1,\gamma(t_1)) > g(t_1,u(t_1,\alpha)) + \eta > g(t_1,u(t_1,\alpha)$$

Therefore,

$$^{C}D_{+}^{\beta}m(t_1,\alpha) > g(t_1,u(t_1,\alpha)) \tag{4.6}$$

From Theorem 4.1, the function $\gamma^*(t)=\gamma(t,t_0,\gamma_0)$ satisfies the IVP (4.3) and the equality

$$\limsup_{h\to 0^+}\tfrac{1}{h^\beta}[\gamma^*(t)-\gamma_0 - S(\gamma^*(t),h)] = f(t,\gamma^*(t)),\ \ holds \tag{4.7}$$

where $\gamma^*(t)=\gamma(t,t_0,\gamma_0)$ is any other solution of (3.5), and

$$S(\gamma^*(t),h) = \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[\gamma^*(t-rh)-\gamma_0\right] \tag{4.8}$$

is the Grunwald Letnikov fractional derivative

Multiply equation (4.7) through by h^β

$$\limsup_{h\to 0^+}[\gamma^*(t)-\gamma_0 - S(\gamma^*(t),h)] = h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t)-\gamma_0 - [S(\gamma^*(t),h)+\rho(h^\beta)] = h^\beta f(t,\gamma^*(t))$$

$$\gamma^*(t) - h^\beta f(t,\gamma^*(t)) = [S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)] \tag{4.9}$$

Then for $t\in[t_0,\infty)$, we have

$$m(t,\alpha)-m(t_0,\alpha)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[m(t-rh,\alpha)-m(t_0,\alpha)\right]=$$

$$\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh,\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right\}$$

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right\}+$$

$$\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\{[\Omega(t-rh),S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)-\Omega(t_0,\gamma_0)]-[\Omega(t-rh,\gamma^*(t-rh))-\Omega(t_0,\gamma_0)]\}$$

(4.10)

Since $\Omega(t,\gamma)$ is locally Lipschitzian with respect to the second variable, we have that,

$$\leq L\left|(-1)^{r+1}\right|\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)[S(\gamma^*(t),h)+\gamma_0+\rho(h^\beta)-\gamma^*(t-rh)]\right\|$$

where $L>0$ is a Lipschitz constant.

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)[S(\gamma^*(t),h)+\rho(h^\beta))-(\gamma^*(t-rh)-\gamma_0)\right\| \qquad (4.11)$$

Using equation (4.8), equation (4.11) becomes,

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)(\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^\beta C_r)[(\gamma^*(t-rh)-\gamma_0)+\rho(h^\beta))-(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)(-1)^{r+1}[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r[(\gamma^*(t-rh)-\gamma_0)]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r\rho(h^\beta)-\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r(\gamma^*(t-rh)-\gamma_0)\right\|$$

$$\leq L(-1)^{r+1}\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}({}^\beta C_r)[(\gamma^*(t-rh)-\gamma_0)]+[\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}{}^\beta C_r\rho(h^\beta)\right\| \qquad (4.12)$$

Substituting equation (4.12) into (4.10) we have

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$

$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^\beta C_r)(\gamma^*(t-rh)-\gamma_0)[\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^\beta)\right\|$$

$$=\Omega(t,\gamma^*(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\left[\Omega(t-rh)-\gamma^*(t)-h^\beta f(t,\gamma^*(t))-\Omega(t_0,\gamma_0)\right]+$$

$$L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^\beta C_r)(\gamma^*(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r\,\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1\,\beta}C_r\rho(h^\beta)\right\|$$

Dividing through by $h^\beta>0$ and taking the $\limsup\ as\ h\to 0^+$ we have,

$$ {}^{C}D_{+}^{\beta}m(t,\alpha)=\limsup_{h\to 0^{+}}\tfrac{1}{h^{\beta}}\left\{\Omega(t,\gamma^{*}(t))-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[\Omega(t-rh)-\gamma^{*}(t)-h^{\beta}f(t,\gamma^{*}(t))-\Omega(t_0,\gamma_0)\right]\right\}+ $$

$$ \limsup_{h\to 0^{+}}\tfrac{1}{h^{\beta}}L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^{*}(t-rh)-\gamma_0)[-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r}\,{}^{\beta}C_r-1]+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\rho(h^{\beta})\right\| $$

Recall that,

$$ \lim_{h\to 0^{+}}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)=-1\ and\ \limsup_{h\to 0^{+}}\tfrac{1}{h^{\beta}}\rho(h^{\beta})=0 $$

From equations (3.6) and (3.7) in [1], we have that

$$ {}^{C}D_{+}^{\beta}m(t,\alpha)={}^{C}D_{+}^{\beta}\Omega(t,\gamma^{*}(t))+L\left\|\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}({}^{\beta}C_r)(\gamma^{*}(t-rh)-\gamma_0)[-(-1)-1]+0\right\| $$

Using condition (ii) of the Theorem 4.1, we obtain the estimate

$$ {}^{C}D_{+}^{\beta}m(t,\alpha)\le g(t,\Omega(t,\gamma^{*}(t))=g(t,m(t,\alpha)), \tag{4.13} $$

Also,

$$ m(t_0^{+},\alpha)\le u_0\ and\ m(t_k^{+},\alpha)=\Omega(t_k^{+},\gamma(t_k)+I_k(\gamma(t_k))\le\rho_k(m(t_k)) \tag{4.14} $$

Now equation (4.14) with $t=t_1$ contradicts (4.6), hence (4.5) is true. □

For $t\in[t_0,T]$, we now establish that

$$ u_{\eta_1}(t,\alpha)<u_{\eta_2}(t,\alpha)\quad \text{whenever } \eta_1<\eta_2 \tag{4.15} $$

Observe that the inequality (4.15) holds for $t=t_0$

Assume that (4.15) is not true. Then there exists a point t_1 such that $u_{\eta_1}(t_1,\alpha)=u_{\eta_2}(t_2,\alpha)$ and $u_{\eta_1}(t,\alpha)<u_{\eta_2}(t,\alpha)\quad for\ t\in[t_0,t_1).$

By Lemma 3.2, we have that

$$ {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha)-u_{\eta_2}(t_2,\alpha))>0 $$

However,

$$ \begin{aligned} {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha)-u_{\eta_2}(t_2,\alpha)) &= {}^{C}D_{+}^{\beta}(u_{\eta_1}(t_1,\alpha)-u_{\eta_2}(t_2,\alpha)) \\ &= g(t_1,u(t_1,\alpha))+\eta_1-[g(t_1,u(t_1,\alpha))+\eta_2] \\ &= \eta_1-\eta_2<\mathbf{0} \end{aligned} $$

which is a contradiction, and so (4.15) is true. Thus, equations (4.5) and (4.15) guarantee that the family of solutions $\{u_{\eta}(t,\alpha)\}$, $t\in[t_0,T]$ of (4.3) is uniformly bounded, i.e. there exists $\lambda>0$ with $|u_{\eta}(t,\alpha)|\le\lambda$, with bound λ on $[t_0,T]$. We now show that the family $\{u_{\eta}(t,\alpha)\}$ is equicontinuous on $[t_0,T]$. Let $k=\sup\{\|g(t,\gamma)|:(t,\gamma)\in[t_0,T]\times[-\lambda,\lambda]\}$,, where λ is the bound on the family $\{u_{\eta}(t,\alpha)\}$. Fix a decreasing sequence $\{\eta_i\}_{i=0}^{\infty}(t)$, such that $\lim_{i\to\infty}\eta_i=0$ and consider a sequence of functions $\{u_{\eta}(t,\alpha)\}$. Again, let $t_1,t_2\in[t_0,T]$, $with\ t_1<t_2$, then we have the following estimates,

$$ \begin{aligned} \|u_{\eta}(t_2,\alpha)-u_{\eta}(t_1,\alpha)\| \le \| & u_0+\eta_i+\frac{1}{\Gamma(\beta)}\int_{t_0}^{t_2}(t_2-s)^{\beta-1}(g(s,u(s,\eta_i))+\eta_i)ds \\ & -\left(u_0+\eta_i+\frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_1-s)^{\beta-1}(g(s,u(s,\eta_i))+\eta_i)ds\right)\| \end{aligned} $$

$$\leq \frac{1}{\Gamma(\beta)}\left\|\left[\int_{t_0}^{t_2}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right](g(s,u(s,\eta_i))u(t_2,\eta_i)ds\right\|$$

$$\leq \frac{k}{\Gamma(\beta)}\left\|\left(\int_{t_0}^{t_2}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right)ds\right\|$$

$$\leq \frac{k}{\Gamma(\beta)}\left\|\left(\int_{t_0}^{t_1}(t_2-s)^{\beta-1}+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}\right)ds\right\|$$

$$\leq \frac{k}{\Gamma(\beta)}\left\|\int_{t_0}^{t_1}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}+\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds\right\|$$

$$\leq \frac{k}{\Gamma(\beta)}\left[\left\|\int_{t_0}^{t_1}(t_2-s)^{\beta-1}-\int_{t_0}^{t_1}(t_1-s)^{\beta-1}ds\right\|+\left\|\int_{t_1}^{t_2}(t_2-s)^{\beta-1}ds\right\|\right]$$

$$\leq \frac{k}{\beta\Gamma(\beta)}\left[\left\|(t_2-t_0)^{\beta}-(t_1-t_0)^{\beta}-(t_2-t_1)^{\beta}\right\|+\left\|(t_2-t_1)^{\beta}\right\|\right]$$

$$\leq \frac{k}{\beta\Gamma(\beta)}\left[(t_2-t_1)^{\beta}+(t_2-t_1)^{\beta}\right]=\frac{2k}{\Gamma(\beta+1)}(t_2-t_1)^{\beta}$$

provided $\|t_2-t_1\|<\delta(\varepsilon)=\left(\frac{\Gamma(\beta+1)\varepsilon}{2k}\right)^{\frac{1}{\beta}}$, proving that the family of solutions $\{u_\eta(t,\alpha)\}$ is equicontinuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t,\alpha)\}$ guarantees the existence of a subsequence $\{u_{\eta_{ij}}(t,\alpha)\}$ which converges uniformly to the function $\vartheta(t)$ on $[t_0,T]$. Then we show that $\vartheta(t)$ is a solution of (4.4). Thus, equation (4.4) becomes

$$u_{\eta_{i_j}}(t,\alpha)=u_{0i_j}+\eta_{i_j}+\tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g_{i_j}(s,u_{\eta_{i_j}}(s,\alpha))+\eta_{i_j})ds,\ t\in[t_0,\infty).$$

(4.16)

Taking the $\lim\ as\ i_j\to\infty$ in (4.16) yields,

$$\vartheta(t)=u_0+\eta_{i_j}+\tfrac{1}{\Gamma(\beta)}\int_{t_0}^{t}(t-s)^{\beta-1}(g(s,\vartheta(t))ds,\ t\in[t_0,\infty). \qquad (4.17)$$

Hence, $\vartheta(t)$ is a solution of (3.5) on $[t_0,T]$. We claim that $\vartheta(t)$ is the maximal solution of (3.5). Then, from (4.5), we have that $m(t,\alpha)<u_\eta(t,\alpha)\leq\vartheta(t)\ on\ [t_0,T]$..

Suppose that in Theorem 4.1, $g(t,u)\equiv 0$, then we have the following results

Corollary 4.1.

Assume that Condition (i) of Theorem 4.1 holds and,

(i) $\Omega\in PC[R_+\times R^N,R_+{}^N]$ and $\Omega\in\chi$ such that

$${}^{C}D_+^{\beta}\Omega(t,\gamma)\leq 0$$

(4.14)

holds, and

$\Omega(t_k,\gamma+I_k(\gamma(t_k)))\leq\omega_k(\Omega(t,\gamma(t))),\ t=t_k,\gamma\in S_\psi$ and the function $\omega_k:R_+^N\to R_+^N$ is nondecreasing for $k=1,2,...$

Then for $t \in [t_0, \infty),$ the inequality

$$\Omega(t, \gamma(t)) \le \Omega(t_0^+, \gamma_0) \text{ holds}$$

V. MAIN RESULTS

In this section, we will obtain sufficient conditions for the practical stability of the system (3.3). Again we assume $0 < \beta < 1.$

Theorem 5.1. Assume that

i) $g \in PC[R_+ \times R_+^n, R^n]$ is piecewise continuous in $(t_{k-1}, t_k]$ and for each $u \in R^n,\ k = 1,2,...,$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,y) = g(t_k^+, u)$ exists, and $g(t,u)$ is quasimonotone nondecreasing in u

ii) $\Omega \in PC[R_+ \times R^N, R_+{}^N]$ and $\Omega \in \chi$ such that ${}^C D_+^\beta \Omega(t,\gamma) \le g(t, \Omega(t,\gamma)), t \ne t_k,\ holds\ \ for\ all\ (t,\gamma) \in R_+ \times S_\Psi,$
There exists $\psi_0 > 0$ such that $\gamma_0 \in S_\psi$ implies that $\gamma(t) + I_k(\gamma(t_k)) \in S_\psi$ and $\Omega(t_k, \gamma + I_k(\gamma(t_k))) \le \omega_k(\Omega(t, \gamma(t))), t = t_k, \gamma \in S_\psi$ and the function $\omega_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

iii) $b(\|\gamma\|) \le \Omega_0(t,\gamma),$ where $b \in K$ and $\Omega_0(t,\gamma), = \sum_{i=1}^N \Omega_i(t,\gamma).$

Then the practical stability of the trivial solution $u = 0$ of (3.5) implies the practical stability of the trivial solution $\gamma = 0$ of (3.3).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (3.5) is stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^N u_{i0} < \delta \text{ implies } \sum_{i=1}^N u_i(t, t_0, u_0) < b(\varepsilon), t \ge t_0 \tag{5.1}$$

where $u(t, t_0, u_0)$ is any solution of (3.5).

Choose $u_0 = \Omega(t_0^+, \gamma_0).$

Since $\Omega(t,x)$ is continuous, then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \delta) > 0$ that is continuous in t_o for each δ such that the inequalities

$$\|\Omega(t,\gamma) - \Omega(t_0, \gamma_0)\| < \delta \text{ implies } \|\gamma - \gamma_0\| < \delta_1$$

and as $\|\Omega(t,\gamma)\| \to 0, \|\gamma\| \to 0$ then the inequalities

$$\|\gamma_0\| < \delta_1 \ \ and\ \ \|\Omega(t_0, \gamma_0) < \delta\| \tag{5.2}$$

are satisfied simultaneously.

We claim that, if $\|\gamma_0\| < \delta_1$ then $\|\gamma(t, t_0, \gamma_0)\| < \varepsilon$.

Suppose that this claim is false, then there would exists a point $t_1 \in [t_0, t)$ and the solution $\gamma(t, t_0, \gamma_0)$ with $\|\gamma_0\| < \delta_1$ such that

$$\|\gamma(t_1)\| = \varepsilon\ \ and\ \ \|\gamma(t)\| < \varepsilon \text{ for } t \in [t_0, t_1)$$

(5.3)

So that using equation (5.3); condition (iii) of Theorem 5.1 reduces to the form

$$b(\|\gamma(t_1)\|) \le \sum_{i=1}^{N} \Omega_i(t_1, \gamma(t_1))\text{, implying}$$

$$b(\varepsilon) \le \sum_{i=1}^{N} \Omega_i(t_1, \gamma(t_1))$$

(5.4)

for $t \in [t_0, t_1)$ and from Theorem 4.1,

$$\Omega(t, \gamma(t) \le \vartheta(t)$$

(5.5)

where $\vartheta(t) = \sum_{i=1}^{n} \vartheta_i(t, t_0, u_0)$ is the maximal solution of (3.5).

Then, using equations (5.4), (5.3), (5.5) and condition (iii) of Theorem 5.1 we arrive at the estimate

$$b(\varepsilon) \le \Omega_0(t_1, \gamma(t_1)) \le \sum_{i=1}^{N} \vartheta_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the practical stability of the trivial solution $u = 0$ of (3.5) implies the practical stability of the trivial solution $\gamma = 0$ of (3.3).

VI. APPLICATION

Consider the system of fractional differential equations

$$^{C}D^{\beta}\gamma_1(t) = -6\gamma_1 - \tfrac{\gamma_2^2 \cos\gamma_1}{2\gamma_1} + \gamma_1 \sin\gamma_2 + \tfrac{\gamma_2^2 \tan\gamma_1}{\gamma_1},\ t \ne t_k$$

$$^{C}D^{\beta}\gamma_2(t) = \tfrac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin\gamma_1 - \gamma_2 \sec\gamma_1 - \gamma_1^2 \cos\gamma_2,\ \ t \ne t_k$$

$$\Delta\gamma_1 = \xi_k(\gamma(t_k)),\ t = t_k$$

$$\Delta\gamma_2 = \tau_k(\gamma(t_k)),\ \ t = t_k\ ,\ k = 1,2,...$$

(6.1) with initial conditions

$$\gamma_1(t_0^+) = \gamma_{10}\ \ and\ \gamma_2(t_0^+) = \gamma_{20}$$

where $\gamma_1, \gamma_2 \in R^N$ are arbitrary functions.

Equation (6.1) is equivalent to (3.3) and $f = (f_1 f_2),$ where $f_1(t, \gamma_1) = {}^{C}D^{\beta}\gamma_1(t) = -6\gamma_1 - \tfrac{\gamma_2^2 \cos\gamma_1}{2\gamma_1} + \gamma_1 \sin\gamma_2 + \tfrac{\gamma_2^2 \tan\gamma_1}{\gamma_1}$ and

$f_2(t, \gamma_2) = {}^{C}D^{\beta}\gamma_2(t)\, \tfrac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin\gamma_1 - \gamma_2 \sec\gamma_1 - \gamma_1^2 \cos\gamma_2$

Consider a vector Lyapunov function of the form $\Omega = (\Omega_1, \Omega_2)^T,$ where

$$\Omega_1(t, \gamma_1, \gamma_2) = \gamma_1{}^2,\ \Omega_2(t, \gamma_1, \gamma_2) = \gamma_2{}^2$$

So that $\Omega = (\Omega_1, \Omega_2)^T$ with $\gamma = (\gamma_1, \gamma_2) \in R^2,$ so that $\|\gamma\| = \sqrt{\gamma^2 + y^2}$

$$\sum_{i=1}^{2} \Omega_i(t, \gamma_1, \gamma_2) = \gamma_1^2 + \gamma_2^2$$

The assumption,

$$b(\|\gamma\|) \le \sum_{i=1}^{n} \Omega_i(\gamma, y) \le a(t, \|\gamma\|)\text{ reduces to}$$

$$\sqrt{\gamma^2 + y^2} \le \gamma^2 + y^2 \le 2\left(\sqrt{\gamma^2 + y^2}\right)^2$$

with the proviso that $b(\wp) = \wp, and\ a(\wp) = 2\wp^2$.

Furthermore, we deduce that using equation (3.4) and $\Omega_1(t, \gamma_1, \gamma_2) = \gamma_1{}^2$

$$^{C}D_{+}^{\beta}\Omega(t,\gamma_1,\gamma_2)=\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\left\{\Omega(t,\gamma_1,\gamma_2)-\Omega(t_0,\gamma_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\,{}^{\beta}C_r\left[\Omega(t-rh,\gamma-h^\beta f_i(t,\gamma_1,\gamma_2))-\Omega(t_0,\gamma_0)\right]\right\},t\ge t_0$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)=\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\left\{\gamma_1^{\ 2}-\gamma_{10}^2)+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)\left[\Omega(t-rh,\gamma-h^\beta f_1(t,\gamma_1))-\gamma_{10}^2\right]\right\},t\ge t_0$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)=\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\left\{\gamma_1^{\ 2}-\gamma_{10}^2)+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)\quad[(\gamma_1-h^\beta f_1(t,\gamma_1))^2-\gamma_{10}^2]\right\},t\ge t_0$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\left\{\gamma_1^{\ 2}-\gamma_{10}^2)+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)[\gamma_1^2-2\gamma_1h^\beta f_1(t,\gamma_1)+h^{2\beta}f_1^{\ 2}(t,\gamma_1))-\gamma_{10}^2]\right\},t\ge t_0$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\left\{\gamma_1^{\ 2}+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)\gamma_1^2-\gamma_{10}^2-+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)\gamma_{10}^2-2\gamma_1\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)h^\beta f_1(t,\gamma_1)\right\},t\ge t_0$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\left\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)\gamma_1^2-\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)\gamma_{10}^2-2\gamma_1\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)h^\beta f_1(t,\gamma_1)\right\},t\ge t_0 \qquad (6.2)$$

Recall from equations (3.7) and (3.8) in [1] that

$$\limsup_{h\to 0^+}\tfrac{1}{h^\beta}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)=\frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)}\ \textit{and}\ \lim_{h\to 0^+}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^{\beta}C_r)=-1 \qquad (6.3)$$

and substituting (6.3) into (6.2), we have,

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)}-\frac{\gamma_{10}^2}{t^\beta\Gamma(1-\beta)}+2\gamma_1 f_1(t,\gamma_1)$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)}+2\gamma_1 f_1(t,\gamma_1)$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)}+2\gamma_1(-6\gamma_1-\tfrac{\gamma_2^2\cos\gamma_1}{2\gamma_1}+\gamma_1\sin\gamma_2+\tfrac{\gamma_2^2\tan\gamma_1}{\gamma_1})$$

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le\frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)}-12\gamma_1^2-\gamma_2^2\cos\gamma_1+2\gamma_1^2\sin\gamma_2+\gamma_2^2\tan\gamma_1$$

$As\ t\to\infty,\ \frac{\gamma_1^2}{t^\beta\Gamma(1-\beta)}\to 0,$ so we now have that

$$\begin{aligned}^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)&\le-12\gamma_1^2-\gamma_2^2\cos\gamma_1+2\gamma_1^2\sin\gamma_2+\gamma_2^2\tan\gamma_1\\&=2\gamma_1^2(-6+\sin\gamma_2)+\gamma_2^2(-\cos\gamma_1+2\tan\gamma)_1\\&\le 2\gamma_1^2(-6+|\sin\gamma_2|)+\gamma_2^2(\tfrac{2\|\sin\gamma_1|}{|\cos\gamma_1|}-|\cos\gamma_1|)\\&\le 2\gamma_1^2(-6+1)+\gamma_2^2(2-1)\\&=\gamma_1^2(-10)+\gamma_2^2(1)\end{aligned}$$

Therefore,

$$^{C}D_{+}^{\beta}\Omega_1(t,\gamma_1)\le-10\Omega_1+\Omega_2$$

(6.4)

Also for $\gamma_0 \in S_\psi, \; for \; t = t_k, \; k = 1,2,...$ we have, $\Omega_1(t, \gamma(t) + \xi_k) = |\xi_k + \gamma(t)| \le \Omega_1(t, \gamma(t))$

Similarly, we compute for the Dini derivative for $\Omega_2(t, \gamma_2) = \gamma_2^2$ and follow through the same argument by substituting for $f_2(t, \gamma_2) = {}^C D^\beta \gamma_2(t) = \frac{3\gamma_1^2}{\gamma_2} - \gamma_2 \sin \gamma_1 - \gamma_2 \sec \gamma_1 - \gamma_1^2 \cos \gamma_2$ in equation (3.4) to have that,

$$ {}^C D_+^\beta \Omega_2(t, \gamma_2) \le 4\Omega_1 - 4\Omega_2 \tag{6.5}$$

Also for $\gamma_0 \in S_\psi, \; for \; t = t_k, \; k = 1,2,...$ we have, $\Omega_2(t, \gamma(t) + \tau_k) = |\tau_k + \gamma(t)| \le \Omega_2(t, \gamma(t))$

By combining equations (6.4) and (6.5), we have

$$ {}^C D_+^\beta \Omega \le \begin{pmatrix} -10 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = g(t, \Omega) \tag{6.6}$$

Now, consider the comparison system ${}^C D^\beta u = g(t, \Omega) = Au$, where $A = \begin{pmatrix} -10 & 1 \\ 4 & -4 \end{pmatrix}$.

The vectorial inequality (6.6) and all other conditions of Theorem 5.1 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (6.1) is practically stable. Therefore, the trivial solution $\gamma = 0$ of the IFrDE (6.1) is practically stable.

VII. CONCLUSION

In this paper, we extend the study on the stability properties of impulsive Caputo fractional differential equations using the scalar Lyapunov function to the vector Lyapunov functions. By adopting the new definition of the Dini derivative as proposed in [1] which is extended to vector, and ogether with the vector impulsive fractional differential equations and comparison results, sufficient conditions for the practical stability of the impulsive Caputo fractional order system (3.3) is established with an illustrative example.

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