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The Functional Dynamics of λ and μ-Pentajection Operators in Topological and Locally Convex Spaces

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Abstract
The research article is "The application of λ and μ- pentajection operators". In this chapter in the theorem I, we have proved that each λ- pentajection operator on a topological linear space can be summing operator. In the theorem II We have proved that each summing operator can be a λ- pentajection operator. In the theorem - III we have proved that each μ- pentajection operator on a locally convex space can be a paracompact operator and a precompactly summing operator. In the theorem IV we have proved that each precompactly summing operator on a locally convex space can be a μ - pentajection operator. In the theorem V we have proved that for each μ- pentajection operator on a locally convex space there can exist that locally convex space as the nuclear locally convex space.

I. Introduction

The research delves into the mathematical framework surrounding the λ- and μ-pentajection operators in topological and locally convex spaces. Through a series of theorems, the study explores the intricate relationships between these operators and various types of summing operators. In Theorem I, it is established that each λ-pentajection operator in a topological linear space can be considered a summing operator. Conversely, Theorem II proves that summing operators can be λ-pentajection operators. Further, Theorem III highlights that each μ-pentajection operator on a locally convex space functions as both a precompact operator and a precompactly summing operator. Theorem IV demonstrates the reverse, proving that precompactly summing operators on locally convex spaces can also be μ-pentajection operators. Lastly, Theorem V reveals that a locally convex space where μ-pentajection operators exist can be classified as a nuclear locally convex space. These findings not only contribute to a deeper understanding of pentajection operators but also expand their potential applications within various branches of functional analysis.

By N we denote the set of natural numbers and by 1 we denote the set of positive integers. By [x_i, I] we denote the family of elements $x_i \in$ E where E is a locally convex space or a topological vector space we denote by [Tx_i, I] as the family of elements Tx $\in$ F for the locally convex space or for the topological vector space F. By a_n we denote the continuous linear forms defined on the space E such that $a_n \in$ E'. By pU (x) we denote the seminorm of the element x $\in$ U where U is a set in any space. By u_f(E) we denote the fundamental system of zero neighbourhoods. By α(E, F) we denote the linear space of continuous linear operators from E into F as spaces.

DEFINITION-1: Let each T$\in$α (E, F) be a continuous linear operator from any space E into the other space F such that

$$x_i \in E \Leftrightarrow \lambda x_i \in E \text{ for } i = 1, 2, 3, 4, 5 \in I \text{ and } \sum_{I} ||T(\lambda x_i)|| \leq \frac{1}{5} \sum_{I} ||\lambda x_i||$$

Where T (λx_i) $\subset$ F with λ < μ

Then each T is said to be a λ-pentajection operator.

DEFINITION II: Let each T$\in$α (E, F) be a continuous linear operator from the topolological linear space E into the topological linear space F such that

$$x_i \in E \Leftrightarrow \mu x_i \in E \text{ for } i = 1, 2, 3, 4, 5 \in I \text{ and } \sum_{I} ||T(\mu x_i)|| \leq \frac{1}{5} \sum_{I} ||\mu x_i||$$

Where T (μx_i) $\in$F with $\mu < \lambda$ Then each T is said to be a μ -pentajection operator.

THEOREM-1: Let there exist each λ- pentajection operator on the topological linear space E. Then this operator can be a summing operator.

PROOF: We can consider that each T$\in\alpha$ (E, F) is a λ- pentajection operator from the topological space E into itself. ………………………(1)

There can be supposed that $x_i \in$E for i = 1, 2, 3, 4, 5 such that $\|x_i\|$ = finite positive numbers. …(2)

From the concepts of (1) and (2) it follows that

$\|Tx_i\|$ = finite positive numbers. with $Tx_i \in$T(E) ………………….(3)

We set T(E) = G such that $Tx_i \in G \subset E$. on the basis of the concept of (1)……………. (4)

On the basis of the concepts of (3) and (4) it is obvious that $Tx_i \in G \subset E$. ……….... (5)

From the concept of (2) it is clear that $\sum\|x_i\| \neq \infty$. …………………... (6)

There can be considered a number $g \in K$

such that $\| g \| = | g |$= a finite positive number. .... (7)

On the basis of the concepts of (6) and (7) it follows

that $\sum || g\,(x_i) || \neq \infty$ for i=1, 2, 3, 4, 5 $\in$I........ (8)

From the concept of (8) it is obvious that $\sum_I || g\,(x_i) || < +\infty$ . . for $x_i \in$E, i$\in$I, $g \in$K....... (9)

We know that $\sum_I || Tx_i || \leq M \sum_I || x_i || \Rightarrow \sum_I || Tx_i|| \leq \frac{1}{5} \sum_I || x_i || \leq \lambda$ ………………..... (10)

On the basis of the concepts of (9) and (10) it is clear that $\sum_I || g\,(x_i) || < +\infty$ …………..... (11)

We set that $\lambda x_i = t_i \in E$. ……….... (12)

From the concepts of (11) and (12) it follows that E

$\sum_I || g\,(t_i) || < +\infty$ for $t_i \in$ E, i = 1, 2, 3, 4, 5 $\in$ I, $g \in$ K. .... (13)

On the basis of the concept of (13) it is obvious that there can exist each summable family $[t_i$,I] of elements from the topological linear space E which can be constructed to be a locally convex space. ……………….(14)

From the concept of (10) it is obvious that

$\sum_I || T(\lambda x_i) || \neq \infty$……………. (15)

On the basis of the concepts of (12) and (15) it is clear that

$\sum_I h || (Tt_i) || < \infty$ for $Tt_i \in G \subset E$, I $\in$ I, $h \in$ K…………………(16)

From the concept of (16) it follows that there can exist each summable family $[Tt_i$,I] of elements from the topological linear space G $\subset$ E which can be constructed to be a locally convex space. On the basis of the concepts of (14) and (17) it is obvious that each T$\in\alpha$(E,F) which is λ-pentajection operator can be a summing operator. Thus, the theorem is completely proved.

THEOREM II: Let each T$\in\alpha$(E,F) be a summing operator. Then each T can be a λ-pentajection operator.

PROOF: We can consider that E and F are locally convex spaces as topological vector spaces.... (1)

Then there can exist each summable family [x_i , I] of elements x_i from E such that $\sum_{I} || g(x_i)$ = a finite positive number for $x_i \in$ E, i ∈ l, $g \in$ K……(2)

We can consider a number λ such that $\sum_{I} || g(\lambda x_i) || \neq \infty$. for $\lambda x_i \in$ E. .... (3)

From the concepts of (2) and (3) it follows that $\sum_{I} || g(\lambda x_i) || < +\infty$ …………….. (4)

On the basis of the concept of (1) it is obvious that there can exist each summable family [Tx_i, 1] of elements from I as each 1 ∈α(E,F) is a summing mapping such that for $\sum_{I} h || (Tx_i) || < + \infty$ for $Tx_i \in$ I, i ∈ I, Where h is a number …………….. (5)

There can be determined the value of λ such that $Tx_i \in$ I $\Leftrightarrow \lambda Tx_i \in$ I, ……….(6)

From the concepts of (5) and (6) it follows that

$\sum_{I} h || (\lambda Tx_i) || \neq +\infty$ such that $\sum_{I} || (\lambda Tx_i) || =$ a finite positive number ……….(7)

On the basis of the concept of (4) it is obvious

that $\sum_{I} h || (\lambda Tx_i) || =$ a finite positive number ……….(8)

From the concepts of (7) and (8) it is clear that

$\sum_{I} || (\lambda Tx_i) || = \sum_{I} || (\lambda x_i) ||$ …………………………………….. (9)

for any positive intger I ∈ 1. On the basis of the concept of (9) it follows that

$\sum_{I} || (\lambda Tx_i) || = M \sum_{I} || (\lambda x_i) ||$

where M is any number for the validity of the given inequality. …. (10)

There can be determined the value of M and the value

of I ∈ I such that $\sum_{I} || (\lambda Tx_i) || \leq \frac{1}{i} \sum_{I} || \lambda x_i ||$ for $x_i \in$ E $\Leftrightarrow \lambda x_i \in$ E and for $Tx_i \in$ F $\Leftrightarrow Tx_i \in$ F…..(11)

We can consider i = 1, 2, 3, 4, 5,.......... …………. (12)

then i = 1, 2, 3, 4, 5, ..., i ∈ l such that i = 1, 2, 3, 4,5 ∈ l ……….... (13)

From the concepts of (11), (12) and (13) it follows that $\sum_{I} || (\lambda Tx_5) || = \frac{1}{5} \sum_{I} || \lambda x_5 ||$ with 5 ∈ l for $\lambda x_5 \in$ E and $\lambda Tx_5 \in$ F .... (14)

On the basis of the concept of (14) it is obvious that each T∈α(E, F) can be λ- pentajection operator.

Thus, the theorem is completely proved.

THEOREM - III: Let each T∈α(E, F) be a μ-pentajection operator from a locally convex space E into a locally convex space F. Then each T can be a precompact operator and a precompactly summing operator.

PROOF: It is obvious that T(E)F such that

$x_i \in$ E $\Leftrightarrow$ $\mu x_i \in$ E and $Tx_i \in$ F $\Leftrightarrow$ $\mu Tx_i \in$ F for i = 1,2,3,4,5 ∈ I. Where E and F can be topological linear spaces. ………………..(1)

On the basis of the concept of (1) it follows that

$\sum_{I} ||\ (Tx_i)|| \le M \sum_{I} ||\ x_i|| \Leftrightarrow \sum_{I} ||\ (\mu Tx_i)|| \le M \sum_{I} ||\ \mu x_i||$

where M is a number $=\frac{1}{5}$, i ∈ l, i = 1, 2, 3, 4, 5. .... (2)

We consider a closed unit ball A in E such that $x_i \in A \Leftrightarrow \mu x_i \in A$ and

$Tx_i \in T(A) \Leftrightarrow \mu Tx_i \in T(A)$ for T∈α(E, F) .... (3)

We know that a closed unit ball in a locally convex space can be totally bounded or precompact. ... (4)

From the concepts of (3) and (4) it is obvious that each A can be totally bounded or precompact. ... (5)

It is known that the image set of a totally bounded set or a precompact set in a locally convex space can be (6) totally bounded or precompact. ……………….. (6)

On the basis of the concepts of (3), (4), (5) and (6) it is clear that each T(A) can be a totally bonded set or a precompact set for $x_i \in A$ and $Tx_i \in T(A)$. .... (7)

From the concepts of (3) and (7) it follows that each T(A) can be totally bounded or precompact when $Tx_i \in T(A)$ and $\mu x_i \in \mu A$ ………………………(8)

On the basis of the concepts of (2) and (8) it is obvious that each μ- pentajection operator T∈α (E, F) can be a precompact operator. …………………..... (9)

On the basis of the concepts of (9) it is clear that

$\sum_{I} ||(Tx_i)|| \neq \infty$ and $\frac{1}{5}\sum_{I} ||(x_i)|| \neq \infty$ such that

$\sum_{I} ||(\mu Tx_i)|| \neq \infty$ and $\frac{1}{5}\sum_{I} ||(\mu x_i)|| \neq \infty$ …………… (10)

From the concept of (10) it follows that $\sum_{I} ||(\mu x_i)|| < +\infty$ for $\mu x_i \in A$ where A is a precompact set on the basis of the concept of (5)……………………… (11)

On the basis of the concept of (11) it is obvious that there can exist each precompactly summable family [ux_i, 1] of the elements from the locally convex space E. …………… (12)

From the concept of (10) it as clear that $\sum_{I} ||(\mu x_i)|| < +\infty \Leftrightarrow \sum_{I} ||T(\mu x_i)|| < +\infty$ for $T(\mu x_i) \in T(A) \subset F$. ………………………….(13)

On the basis of the concept of (13) it follows that there can exist each precompactly summable family [Tμx_i, 1] from F for $T\mu x_i \in T(A) \subset F$………………………..(14).

From the concepts of (12) and (14) it is obvious that each μ-pentajection operator can be a precompactly summing operator. …………………………………….(15)

On the basis of the concepts of (9) and (15) it is clear that each μ-pentajection operator can be a precompact operator and a precompactly summing operator.

Thus the theorem is completely proved.

THEOREM - IV: Let each T∈α (E, F) be a precompactly summing operator where E and F are locally convex spaces. Then each T can be a μ -pentajection operator.

PROOF: It is obvious that T(E) ⊂ F such that $Tx_i \in F$ for $x_i \in E$, i ∈ l. ………….. (1)

We can consider a precompact set D in the locally convex space E such that T(D) can be a precompact set in the locally convex space F as the image set of the precompact set is a precompact set. ……….(2)

From the concepts of (1) and (2) it follows that

$x_i \in E \Leftrightarrow x_i \in D$ such that $\mu x_i \in E \Leftrightarrow \mu x_i \in D$ and $Tx_i \in T(D) \subset F \Leftrightarrow T\,\mu x_i \in T(D) \subset F$…………. (3)

On the basis of the concept of (3) it is obvious that there can exist each precompactly summable family $[x_i, I]$ of elements from

such that $\sum_I ||(x_i)|| \neq \infty$ for $x_i \in D$, $i \in I$. …………………….(4)

Moreover, there can exist each precompactly summable family [Tx_i, I] If of elements from such that $\sum_I ||(Tx_i)|| < +\infty$ ……………………..(5)

From the concept of (3) and (4) it is clear that

$\sum_I ||(\mu x_i)|| \neq \infty$ for $\mu x_i \in D \subset E$, ………………..(6)

On the basis of the concepts of (3) and (5) it follows

that $\sum_I ||(T\mu x_i)|| \neq \infty$ for $T\mu x_i \in D \subset F$, ………………(7)

From the concept of (6) it is obvious that

$\sum_I ||(\mu x_i)|| < +\infty$ for $\mu x_i \in D \subset E$, ………………………..(8)

On the basis of the concept of (7) it is clear that

$\sum_I ||(T\mu x_i)|| < +\infty$ for $T\mu x_i \in D \subset F$ ……………….. (9)

from the concepts of (8) and (9) it follows that

$\sum_I ||(T\mu x_i)|| < \sum_I ||(\mu x_i)||$ for $\mu x_i \in E$ $T\mu x_i \in F$ …………………(10)

There can exist a number q such that $\sum_I ||(T\mu x_i)|| \leq q \sum_I ||(\mu x_i)||$ …………………(11)

We can select q such that $q = \frac{1}{5}$ …………………….... (12)

From the concepts of (11) and (12) it is obvious that $\sum_I ||(T\mu x_i)|| \leq \frac{1}{5} \sum_I ||(\mu x_i)||$

for $\mu x_i \in E$ $T\mu x_i \in F$ …………….... (13)

On the basis of the concept of (13) it is clear that each T can be a μ- pentajection operator.

Thus, the theorem is completely proved.

THEOREM-V: For each μ -pentajection operator on a locally convex space E. There can exist E as a nuclear locally convex space.

PROOF: It is obvious that each $T \in \alpha(E,F)$ is a μ-pentajection operator on the locally convex space E. such that $T(E) \subset E \Leftrightarrow x_i \in E \Leftrightarrow Tx_i \in T(E)$.

Moreover. $\sum_I ||(T\mu x_i)|| \leq \frac{1}{5} \sum_I ||(\mu x_i)|| \neq \infty$

for $\in E \Leftrightarrow \mu x_i \in E$ and

$Tx_i \in T(E) \Leftrightarrow T\mu x_i \in T(E) \subset E$. …………….... (2)

There can exist a fundamental system $u_f(E)$ of zero neighbourhoods U_n, in the locally convex space E.. …………………………….(3)

We consider that U and V are two zero neighbourhoods

$\in u_f$(E) with V <U. ………....... (4)

On the basis of the concepts of (1) and (2) it follows

that pU(x) =$||x||$ = a finite positive number where we consider

that $\mu x_i \in$ U $\in u_f$(E) for i = 1, 2, 3, 4, 5 ∈ l. ……………...... (5)

We put $T\mu x_i = Tx \in$ V $\in$ T(E) ⊂ E. ……………...... (6)

There can exist each continuous linear form and defined on E such that a_n ε V° ⊂ E'. ………(7)

We know that $\sum_N (x\, a_n) < \sum_N ||x||\ ||a_n||$

such that $\sum_N ||x||\ ||a_n|| =$ a finite positive number. ... (8)

On the basis of the concept of (8) it is obvious

that $\sum_N (x\, a_n)$ can be a finite positive number......... (9)

From the concepts of (5) and (8) it is clear that

$||x|| < \sum_N ||x||\ ||a_n||$ such that $\sum_N (x\, a_n)$ for $x \in$ U $\in u_f$(E), $a_n \in$ V° ⊂ T(E)………………. (10)

There can be considered a number $g \in$K such that $||x|| \leq \sum_N (x, g a_n)$ …………. (11)

We set $g a_n$ = $b_n \in$ V° ⊂ {T(E)}' , $||x|| =$ pU(x). …………………………… (12)

On the basis of the concepts of (11) and (12) it follows

that pU(x)$\leq \sum_N (b_n)$ for $x \in$ U $\in u_f$(E), $b_n \in$ V° ⊂ {T(E)}' ⊂ E' …………………… (13)

From the concepts of (1), (2), (6) and (13) it is obvious that

$||x|| < ||x|| \sum_N ||b_n||$ such that

$\sum_N ||b_n|| \neq \infty$ for $b_n \in$ V° ⊂ {T(E)}' ⊂ E' …………………………… (14)

We can set $||b_n|| =$ p' V°(b_n). …………………………… (15)

on the basis of the concepts of (14) and (15) it is clear

that $\sum_N$ p' V°(b_n) $\neq \infty$ …………………………… (16)

From the concept of (16) it follows that

$\sum_N$ p' V°(b_n) $< + \infty$ for $b_n \in$ V° ⊂ {T(E)}' ⊂ E' …………. (17)

On the basis of the concepts of (3), (4), (5), (6), (13) and (17) it is obvious that the locally convex E can be a nuclear locally convex space.

Thus, the theorem is completely proved.

II. Conclusion

The research on λ and μ-pentajection operators provides significant insight into the functional properties of these operators in topological and locally convex spaces. By proving theorems that connect pentajection operators with summing and precompactly summing operators, the study advances the theoretical framework of operator theory in mathematical analysis. The identification of locally convex spaces as potential nuclear spaces when μ-pentajection operators are present further enhances our comprehension of these spaces' structure. This exploration of pentajection operators opens doors to future studies, particularly in expanding their applications in more complex mathematical systems and spaces.

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