

## Inverse Problem for Singular Sturm-Liouville operator

\*Mehmet Kayalar

Vocational high school, Erzincan University, Erzincan, 24100, Turkey  
 Corresponding Author: Mehmet Kayalar

**Abstract:** In this article, the inverse problem for the singular Sturm-Liouville operator is studied. A set of values of eigenfunctions in some internal point and spectrum are given as a data. Uniqueness theorems are also obtained.

Date of Submission: 17-07-2017

Date of acceptance: 05-08-2017

### I. Introduction

Inverse spectral analysis has been an important research topic in mathematical physics. Inverse problems of spectral analysis involve reconstruction of a linear operator from its spectral characteristics [1,3-9]. For inverse Sturm-Liouville problems, such characteristics are two spectra for different boundary conditions, one spectrum and normalizing constants, spectral functions, scattering data, Weyl function [1,3-18]. An early important result in this direction, which gave vital imputes for the further development of inverse problem theory, was obtained in [1]. Inverse problem for interior spectral data of the differential operator lies in reconstructing this operator by some eigenvalues and information on eigenfunctions at some an internal point in the interval considered. The similar problems for the Sturm-Liouville and diffusion operator was studied in [19-25].

Consider the following singular Sturm-Liouville operator  $L$  satisfying (1)-(3)

$$Ly = -y'' + \left[ q(x) - \frac{1}{4\sin^2 x} \right] y = \lambda y \quad (1)$$

with boundary conditions

$$y(0) = 0, \quad (2)$$

$$y'(\pi - \varepsilon, \lambda) + Hy(\pi - \varepsilon, \lambda) = 0, \quad (3)$$

where  $q(x)$  is summable,  $\varepsilon$  any positive number,  $H$  finite real number and  $\lambda$  spectral parameter. The operator  $L$  is self adjoint on the  $L_2(0, \pi)$  and with (2), (3) boundary conditions has a discrete spectrum  $\{\lambda_n\}$ .

Let us introduce the second singular Sturm-Liouville operator  $L$  satisfying

$$Ly = -y'' + \left[ q(x) - \frac{1}{4\sin^2 x} \right] y = \lambda y \quad (4)$$

subject to the same boundary conditions (2), (3) where  $q(x)$  is summable. The operator  $L$  is self adjoint on the  $L_2(0, \pi)$  and with (2), (3) boundary conditions has a discrete spectrum  $\{\lambda_n\}$ .

### II. Main Results

The Legendre equation is

$$(1-t^2)y'' - 2ty' + n(n+1)y = 0.$$

First, we let

$$t = \cos x, \quad z = y(t) \text{ and } y = z \sqrt{\sin x}$$

then  $y$  satisfies

$$-y'' - \left[ \left( \lambda + \frac{1}{4} \right) + \frac{1}{4\sin^2 x} \right] y = \lambda y,$$

where  $\lambda = n(n+1)$ . For  $0 < \varepsilon < x \leq \pi - \varepsilon < \pi$  and  $n$  sufficiently large, we conclude that Legendre functions are [],

$$P_n(x) = \sqrt{\frac{2}{\pi n}} \cos \left[ \left( n + \frac{1}{2} \right) x - \frac{\pi}{4} \right].$$

Solution of the equation (1)-(3) satisfying  $\varphi(0, \lambda) = 0$ ,  $\varphi'(0, \lambda) = 1$  boundary conditions is

$$\varphi(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda} (x-t) \left[ q(t) - \frac{1}{4 \sin^2 t} \right] \varphi(t, \lambda) dt. \quad (5)$$

Eigenvalues of the problem (1)-(3) are the roots of the (3). These spectral characteristics and eigenfunctions satisfy the following asymptotic expression, respectively

$$\rho_n = \sqrt{\lambda_n} = n + \frac{1}{2} + \frac{c}{\pi \left( n + \frac{1}{2} \right)} + O \left( \frac{1}{n^2} \right), \quad (6)$$

$$\varphi(x, \lambda_n) = \frac{1}{n + \frac{1}{2}} \sin \left( n + \frac{1}{2} \right) x + O \left( \frac{1}{n^2} \right) \quad (7)$$

where  $c = \frac{1}{\pi} \left( H + \frac{1}{2} \int_0^\pi q(t) dt \right)$ .

When  $b = \frac{\pi}{2}$ , we get the following uniqueness **Theorem Error! Reference source not found.**

**Theorem 2.1** *If for every  $n \in \mathbb{N}$  we have*

$$\lambda_n = \lambda_n', \quad \frac{y_n'(\frac{\pi}{2})}{y_n(\frac{\pi}{2})} = \frac{y_n'(\frac{\pi}{2})'}{y_n(\frac{\pi}{2})'} \quad (8)$$

then

$$q(x) = q(x) \text{ a.e on the interval } (0, \pi).$$

In the case  $b \neq \frac{\pi}{2}$ , the uniqueness of  $q(x)$  can be proved if we require the knowledge of a part of the second spectrum.

Let  $m(n)$  be a sequence of natural numbers with a property

$$m(n) = \frac{n}{\sigma} (1 + \varepsilon_n), \quad 0 < \sigma \leq 1, \varepsilon_n \rightarrow 0. \quad (9)$$

**Lemma 2.1** *Let  $m(n)$  be a sequence of natural numbers satisfying (9) and  $b \in (0, \frac{\pi}{2})$  are so chosen that  $\sigma > \frac{2b}{\pi}$ . If for any  $n \in \mathbb{N}$*

$$\lambda_{m(n)} = \lambda_{m(n)}', \quad \frac{y_{m(n)}'(b)}{y_{m(n)}(b)} = \frac{y_{m(n)}'(b)'}{y_{m(n)}(b)'} \quad (10)$$

then  $q(x) = q(x)$  a.e on **Error!**

Let  $l(n)$  and  $r(n)$  be a sequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1} (1 + \varepsilon_{1,n}), \quad 0 < \sigma_1 \leq 1, \varepsilon_{1,n} \rightarrow 0 \quad (11)$$

$$r(n) = \frac{n}{\sigma_2} (1 + \varepsilon_{2,n}), \quad 0 < \sigma_2 \leq 1, \varepsilon_{2,n} \rightarrow 0 \quad (12)$$

and let  $\mu_n$  be the eigenvalues of the problem (1), (2) and (13) and  $\mu_n$  be the eigenvalues of the problem (4), (2) and (13)

$$y'(\pi-\varepsilon, \lambda) + Hy(\pi-\varepsilon, \lambda) = 0. \tag{13}$$

Using Mochizuki and Trooshin's method from Lemma and Theorem **Error! Reference source not found.**, we will prove that the following Theorem holds.

**Theorem 2.2** Let  $l(n)$  and  $r(n)$  be a sequence of natural numbers satisfying (11) and (12), and  $\frac{\pi}{2} < b < \pi$  are so chosen that  $\sigma_1 > \frac{2b}{\pi} - 1, \sigma_2 > 2 - \frac{2b}{\pi}$ . If for any  $n \in \mathbb{N}$  we have

$$\lambda_n = \lambda_n, \mu_{l(n)} = \mu_{l(n)} \text{ and } \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} \tag{14}$$

then

$$q(x) = q(x) \text{ a.e. on } (0, \pi).$$

### III. Proof of the Main Results

**Proof of Theorem Error! Reference source not found.** Before proving the Theorem **Error! Reference source not found.**, we will mention some results, which will be needed later. We get the initial value problems

$$-y'' + \left[ q(x) - \frac{1}{4\sin^2 x} \right] y = \lambda y, \tag{15}$$

$$y(0) = 0, \tag{16}$$

and

$$-y'' + \left[ q(x) - \frac{1}{4\sin^2 x} \right] y = \lambda y, \tag{17}$$

$$y(0) = 0. \tag{18}$$

It can be shown that there exists a kernel  $K(x, t)$  ( $K(x, t)$ ) continuous on  $(0, \pi) \times (0, \pi)$  such that by using the transformation operator every solution of equations (15), (16) and (17), (18) can be expressed in the form

$$y(x, \lambda) \neq \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x K(x, t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt + O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right), \tag{19}$$

$$y(x, \lambda) \neq \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x K(x, t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt + O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right), \tag{20}$$

respectively, where  $\tau = |\operatorname{Im} \sqrt{\lambda}|$  and the kernel  $K(x, t)$  ( $K(x, t)$ ) is the solution of the equation

$$\frac{\partial^2 K(x, t)}{\partial x^2} + \left( \frac{1}{4\sin^2 x} - q(x) \right) K(x, t) \neq \frac{\partial^2 K(x, t)}{\partial t^2} + \frac{1}{4\sin^2 t} K(x, t)$$

subject to the boundary conditions

$$2 \frac{dK(x, x)}{dx} = q(x),$$

$$K(x, 0) \neq 0.$$

Multiplying (15) by  $y(x, \lambda)$  and (17) by  $y(x, \lambda)$  subtracting and integrating from 0 to  $\frac{\pi}{2}$ , we obtain

$$\int_0^{\frac{\pi}{2}} (q(x) - q(x)) y(x, \lambda) y(x, \lambda) dx = \left( y(x, \lambda) y'(x, \lambda) - y(x, \lambda) y'(x, \lambda) \Big|_0^{\frac{\pi}{2}} \right) \tag{21}$$

The functions  $y(x, \lambda)$  and  $y(x, \lambda)$  satisfy the same initial conditions (16) and (18), i.e.,

$$y(0,\lambda)y'(0,\lambda)-y(0,\lambda)y'(0,\lambda)=0.$$

Let

$$Q(x) = q(x) - q(x), \tag{22}$$

$$H(\lambda) = \int_0^{\pi} Q(x) y(x, \lambda) y'(x, \lambda) dx. \tag{23}$$

If the properties of  $y(x, \lambda)$  and  $y'(x, \lambda)$  are considered, the function  $H(\lambda)$  is an entire function.

Therefore the condition of the Theorem **Error! Reference source not found.** imply,

$$y\left(\frac{\pi}{2}, \lambda_n\right) y'\left(\frac{\pi}{2}, \lambda_n\right) - y\left(\frac{\pi}{2}, \lambda_n\right) y'\left(\frac{\pi}{2}, \lambda_n\right) = 0,$$

and hence

$$H(\lambda_n) = 0, n \in N.$$

In addition, using (19) and (23) for  $0 < x < \pi$ ,

$$|H(\lambda)| \leq M \frac{\pi}{2} |b|, b = \text{Im} \sqrt{\lambda}, \tag{24}$$

where  $M$  is constant.

Introduce the function

$$\omega(\lambda) = y'(\pi - \varepsilon, \lambda) + H y(\pi - \varepsilon, \lambda). \tag{25}$$

The zeros of  $\omega(\lambda)$  are the eigenvalues of  $L$  and hence it has only simple zeros  $\lambda_n$ , because of the separated boundary conditions. By using the asymptotic forms of  $y(x, \lambda)$  and  $y'(x, \lambda)$ , we obtain

$$\omega(\lambda) = \cos\left(n + \frac{1}{2}\right)\pi + O\left(\frac{1}{n}\right). \tag{26}$$

From 26,  $\omega(\lambda)$  is an entire function of order  $\frac{1}{2}$  of  $\lambda$ . Since the set of zeros of the entire function  $\omega(\lambda)$  is contained in the set of zeros  $H(\lambda)$ , we see that the function

$$\Psi(\lambda) = \frac{H(\lambda)}{\omega(\lambda)} \tag{27}$$

is an entire function. From (24) (26) and (27), we get

$$|\Psi(\lambda)| = O\left(\frac{1}{\sqrt{\lambda}}\right).$$

So, for all  $\lambda$ , from the Liouville theorem,

$$\Psi(\lambda) = 0,$$

or

$$H(\lambda) = 0.$$

It was proved in [] that there exists absolutely continuous function  $K(x, t)$  such that we have

$$y(x, \lambda) y'(x, \lambda) \mp \frac{1}{2\lambda} \left[ 1 - \cos(2\sqrt{\lambda}x) + \int_0^x K(x, t) \cos(2\sqrt{\lambda}t) dt \right] O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right) \tag{28}$$

where

$$K(x,t) \neq 2 \left[ K(x,x-2t) K(x,x-2t) \right] \int_{-x+2t}^x K(x,s) K(x,s-2t) ds + \int_{-x}^{x-2t} K(x,s) K(x,s+2t) ds .$$

We are now going to show that  $Q(x)=0$  a.e. on **Error!** From (23) and (28), we have

$$\int_0^{\pi} \frac{\pi}{2} Q(x) \left[ 1 - \cos(2\sqrt{\lambda}x) + \int_0^x K(x,t) \cos(2\sqrt{\lambda}t) dt \right] dx + O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right) = 0$$

This can be written as

$$\int_0^{\pi} \frac{\pi}{2} Q(x) dx - \int_0^{\pi} \frac{\pi}{2} \cos(2\sqrt{\lambda}t) \left[ Q(t) + \int_t^{\pi} \frac{\pi}{2} Q(x) K(x,t) dx \right] dt + O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right) = 0$$

Let  $\lambda \rightarrow \infty$  along the real axis, By the Riemann-Lebesgue lemma, we should have

$$\int_0^{\pi} \frac{\pi}{2} Q(x) dx = 0 \tag{29}$$

and

$$\int_0^{\pi} \frac{\pi}{2} \cos(2\sqrt{\lambda}t) \left[ Q(t) + \int_t^{\pi} \frac{\pi}{2} Q(x) K(x,t) dx \right] dt = 0 \tag{30}$$

Thus from the completeness of the functions  $\cos(2\sqrt{\lambda}t)$ , it follows that

$$Q(t) + \int_t^{\pi} \frac{\pi}{2} Q(x) K(x,t) dx = 0, 0 < t < \frac{\pi}{2}. \tag{31}$$

Thus we have obtained

$$Q(x) = q(x) - q(x) = 0$$

or

$$q(x) = q(x)$$

almost everywhere on **Error!**

To prove that  $q(x)=0$  on  $\left[\frac{\pi}{2}, \pi\right]$  almost everywhere, we should repeat arguments for the supplementary problem

$$Ly = -y'' + \left[ q(\pi-x) - \frac{1}{4\sin^2 x} \right] y, \quad 0 < x < \pi$$

subject to the boundary conditions

$$y(\pi-\varepsilon, \lambda) = 0,$$

$$y'(0, \lambda) + Hy(0, \lambda) = 0.$$

Consequently

$$q(x) = q(x) \text{ a.e. on the interval } (0, \pi)$$

Therefore Theorem **Error! Reference source not found.** is proved.

**Proof of Lemma .** As in the proof of Theorem **Error! Reference source not found.**, we can show that

$$G(\rho) = \int_0^b Q(x) y(x, \lambda) y'(x, \lambda) dx = \left( y(x, \lambda) y'(x, \lambda) - y'(x, \lambda) y(x, \lambda) \Big|_{x=b} \right) \tag{32}$$

where  $\rho = \sqrt{\lambda}$  and  $Q(x) = q(x) - q(x)$ . From the assumption

$$\frac{y'_{m(n)}(b)}{y_{m(n)}(b)} = \frac{y'_{m(n)}(b)}{y_{m(n)}(b)}$$

together with the initial condition at 0 it follows that,

$$G(\rho_{m(n)})=0, n \in N.$$

Next, we will show that  $G(\rho)=0$  on the whole  $\rho$  plane.

From (32), we see that the entire function  $G(\rho)$  is a function of exponential type  $\leq 2b$ . One has

$$|G(\rho)| \leq M e^{2br |\sin\theta|} \tag{33}$$

where  $M$  is positive constant,  $\rho = \sqrt{\lambda} = r e^{i\theta}$ .

Define the indicator of function  $G(\rho)$  by;

$$h(\theta) = \lim_{r \rightarrow \infty} \sup \frac{\ln |G(r e^{i\theta})|}{r}. \tag{34}$$

Since  $|Im \sqrt{\lambda}| = r |\sin\theta|$ ,  $\theta = \arg \sqrt{\lambda}$  from (33) and (34) it follows that

$$h(\theta) \leq 2b |\sin\theta|. \tag{35}$$

According to the [ ] set of zeros of every entire function of the exponential type, not identically zero, satisfies the inequality:

$$\lim_{r \rightarrow \infty} \inf \frac{n(r)}{r} \leq \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta \tag{36}$$

where  $n(r)$  is the number of zeros of  $G(\rho)$  in the disk  $|\rho| \leq r$ . By (35),

$$\frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta \leq \frac{b}{\pi} \int_0^{2\pi} |\sin\theta| d\theta = \frac{4b}{\pi}.$$

From the assumption and the known asymptotic expression (6) of the eigenvalues  $\sqrt{\lambda_n}$  we obtain;

$$n(r) \geq \sum_{|\rho| \leq r} 2\sigma r (1 + O(1)),$$

$$\frac{n}{\sigma} \left[ 1 + O\left(\frac{1}{n}\right) \right] < r$$

$r \rightarrow \infty$ .

For the case  $\sigma > \frac{2b}{\pi}$ ,

$$\lim_{r \rightarrow \infty} \frac{n(r)}{r} \geq 2\sigma > \frac{4b}{\pi} = \frac{2b}{\pi} \int_0^{2\pi} |\sin\theta| d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta. \tag{37}$$

The inequalities (36) and (37) imply that  $G(\rho)=0$  on the whole  $\rho$  plane.

Similar to the proof of the Theorem **Error! Reference source not found.**, we have

**Error!**

This completes the proof of Lemma .

**Proof of Theorem .** From

$$\lambda_{r(n)} = \lambda_{r(n)}, \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{y'_{r(n)}(b)}{y_{r(n)}(b)}$$

where  $r(n)$  satisfies (12) and  $\sigma_2 > 2 - \frac{2b}{\pi}$  according to Lemma we get

$$q(x) = q(x) \text{ a.e on } [b, \pi]. \tag{38}$$

Thus, it needs to be proved that  $q(x) = q(x)$  a.e on **Error! Reference source not found.** The eigenfunctions **Error! Reference source not found.** and **Error! Reference source not found.** satisfy the same boundary condition at  $\pi$ . It means that

$$y_n(x, \lambda_n) = \xi_n y_n(x, \lambda_n) \tag{39}$$

on  $(b, \pi)$  for any  $n \in N$  where  $\xi_n$  are constants.

From (32) and (39), we get that;

$$G(\rho) = 0, \text{ for } \rho^2 = \lambda_n, n \in N,$$

and

$$G(\rho) = 0, \text{ for } \rho^2 = \mu_{l(n)}, n \in N,$$

where  $\lambda_n$  and  $\mu_{l(n)}$  satisfy (6).

Counting the number of  $\lambda_n$  and  $\mu_{l(n)}$  located inside the disc of radius  $r$ , we have

$$1 + 2r \left[ 1 + O\left(\frac{1}{n}\right) \right]$$

of  $\lambda_n$ 's and

$$1 + 2r\sigma_1 \left[ 1 + O\left(\frac{1}{n}\right) \right]$$

of  $\mu_{l(n)}$ 's.

This means that

$$n(r) = 2 + 2 \left[ r(\sigma_1 + 1) + O\left(\frac{1}{n}\right) \right]$$

and

$$\lim_{r \rightarrow \infty} \frac{n(r)}{r} = 2(\sigma_1 + 1).$$

Repeating the last part of the proof of Lemma, and considering the condition  $\sigma_1 > \frac{2b}{\pi} - 1$ , we can demonstrate that  $G(\rho) = 0$  identically on the whole  $\rho$ -plane which implies that

**Error!**

and consequently

$$q(x) = q(x) \text{ a.e. on } (0, \pi)$$

### References

- [1] G. Borg, Eine Umkehrang der Sturm-Liouvilleschen Eigenwertaufgabe, Acta Math. 78 (1946), 1-96.
- [2] V.A. Ambarzumyan, Über eine frage der eigenwerttheorie, Z. physics. 53 (1929) 690-695.
- [3] I.M. Gelfand and B.M. Levitan, On the determination of a differential equation from its spectral function, Izv. Akad. Nauk SSR. Ser. Mat. 15 (1951), 309-360 (in Russian); English transl. in Amer. Math. Soc. Transl. Ser. 2 (1) (1955), 253-304.
- [4] F. Gesztesy and B. Simon, Inverse spectral analysis with partial information on the potential II, The case of discrete spectrum, Trans. Amer. Math. Soc. 352(6) (2000), 2765-2787.
- [5] R.P. Gilbert, A method of ascent for solving boundary value problems, Bull. Amer. math. Soc. 75 (1969), 1286-1289.
- [6] B.M. Levitan, On the determination of the Sturm-Liouville operator from one and two spectra, Math. USSR Izv. 12 (1978), 179-193.
- [7] B.M. Levitan, Inverse Sturm-Liouville Problems, VNU Science Press, Utrecht, 1987.
- [8] V.A. Marchenko, Sturm-Liouville Operators and Their Applications, Naukova Dumka, Kiev, 1977; English transl.: Birkhauser, 1986.
- [9] J. Pöschel, E. Trubowitz, Inverse Spectral Theory, Academic Press, Orlando, 1987.
- [10] O.H. Hald, Discontinuous inverse eigenvalue problem, Comm. Pure Appl. Math. 37 (1984) 539-577.
- [11] H. Hochstadt, The inverse Sturm-Liouville problem, Comm. Pure Appl. Math. 27 (1973) 715-729.
- [12] H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, SIAM J. Appl. Math. 34 (1978), 676-680.
- [13] O.R. Hryniv, Y.V. Mykytyuk, Half inverse spectral problems for Sturm-Liouville operators with singular potentials, Inverse problems 20 (2004) 1423-1444.
- [14] E.I. Isaacson, E. Trubowitz, The inverse Sturm-Liouville problem, Comm. Pure Appl. Math. 36 (1978) 767-783.
- [15] W. Rundell, P.E. Sacks, Reconstruction of a radially symmetric potential from two spectral sequences, J. Math. Anal. Appl. 264 (2001) 354-381.
- [16] M.G. Gasymov, On the determination of the Sturm-Liouville equation with peculiarity in zero on two spectrum, DAN SSSR, 161 (1965), 271-276.
- [17] I. Sakhnovich, Half inverse problems on the finite interval, Inverse Problems 17 (2001) 527-532.
- [18] H. Koyunbakan and E.S. Panakhov, Half inverse problem for operators with singular potential, Integral Transforms and Special Functions, 18 (2007), 765-770.
- [19] K. Mochizuki and I. Trooshin, Inverse problem for interior spectral data of Sturm-Liouville operator, J. Inverse Ill-Posed Problems 9 (2001), 425-433.

- [20] C.F. Yang and X.P. Yang, An interior inverse problem for the Sturm-Liouville operator with discontinuous conditions, *Appl. Math. Lett.* (2009), doi:10.1016/j.aml.2008.12.001.
- [21] Y. P. Wang, An interior inverse problem for Sturm-Liouville operators with eigenparameter dependent boundary conditions, *Tamkang Journal of Math.* 42 (2011), 395-403.
- [22] Murat Sat and Etibar S. Panakhov, A uniqueness theorem for Bessel operator from interior spectral data, *Abstract and Applied Analysis*. Vol. 2013. Hindawi Publishing Corporation, 2013.
- [23] E. Panakhov and M. Sat, Inverse problem for the interior spectral data of the equation of hydrogen atom, *Ukrainian Mathematical Journal* 64.11 (2013).
- [24] Chuan-Fu Yang and Yong-Xia Guo, Determination of a differential pencil from interior spectral data, *Journal of Mathematical Analysis and Applications* 375.1 (2011): 284-293.
- [25] Chuan-Fu Yang, An interior inverse problem for discontinuous boundary-value problems, *Integral Equations and Operator Theory* 65.4 (2009): 593-604.
- [26] B. Ja. Levin, *Distribution of zeros of entire functions*, AMS Transl. 5 Providence, (1964).