An Evaluation of Magnetic Field and Density Dependence of the Josephson Frequency of the Coherent Atom-Molecule in Linear Approximation

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Abstract: *In this paper we have presented the method of evaluation of magnetic field and density dependence of the Josephson frequency of the coherent atom-molecule collisions in the linear approximation. Actually linearized time dependent field equation are coupled equation and these equations describe exactly the Rabi oscillation as the coupled harmonic equation with the coupling constant. In the context of particle number oscillations between condensates, Rabi oscillations are referred Josephson oscillations and the associated frequency is called Josephson frequency. This is Josephson oscillation between and atomic condensate and a condensate of bare molecule instead of dressed molecules. Now if one expand the propagation of the molecules around the pole at the bound state energy then one obtains the Josephson oscillations of an atomic and a dressed molecular condensate and the Josephson frequency is obtained. The result agrees well with the experimental fact. Our theoretical result are in good agreement with that of the experimental data. Our theoretical result indicates that Josephson frequency is almost constant for the condensate density but it rises very sharply for condensate density as a function of magnetic field. As expected, the difference between the Josephson frequency and the molecular binding energy increases with increasing condensate density Moreover, for values of the magnetic field closer to resonance the difference is also larger. It gives a great deal of insight in the coherent atom-molecule oscillations and in particular, in their many-body aspects.*

Key words: *Josephson frequency, Rabi oscillation, coupling constant, condensates, bound state energy, resonance.*

I. Introduction

we have studied the coherent atom-molecules oscillations. The experimental observation of atom-molecule coherence in Bose-Einstein condensate has been made [1,2] and its theoretical description has been derived in terms of mean field theory. In the experiments[1,2] performed both in Wieman's group at JILA, one makes use of Feshbach resonance[3] at $B_o = (54.4)$ Gauss in the $(f = 2; m_f = -2)$ hyperfine state of $^{85}R_b$. The width of the resonance is equal to $\Delta B = 11.04$ G and the off resonant background scattering length is given by $a_{bg} = -443$ a_0 where a_0 is the Bohr radius. The difference in the magnetic moment between the open channel and the closed channel is given by $\Delta \mu = -2.23$ μ_B where μ_B is the Bohr magneton.[4] In both experiments, one starts from a stable and essentially pure condensate of about $N_c = 10,000$ atoms at a magnetic field such that the effective scattering length is close to zero. This implies that, since the condensate is in the noninteracting limit, its density profile is detuned by the harmonic oscillator ground state wave function. The harmonic external trapping potential is axially symmetric with trapping frequencies $V_r = 17.4$ and $V_z = 6.8$ Hz in the radial and axial direction respectively. Starting from this situation, one quickly ramps the magnetic field to a value B_{hold} closeto the resonant value and keeps it therefore a short time t_{hold} before ramping to a value E_{evolve}. The magnetic field is kept at this last value for a time t_{evolve} before performing a similar pulse to go back to the initial situation. The duration of all four magnetic field ramps is given by t_{ramp}. Both the r_{amp} time t_{ramp} and the hold time t_{hold} are kept fixed at values of 10-15μs. The time t_{evolve} between the pulses is variable.

Such a double pulse experiment is generally called a Ramsey experiment.[5] Its significance is most easily understood from single system of two coupled harmonic oscillator. The atomic condensate corresponding to oscillator 'a' and the molecular condensate to oscillator 'b'. Therefore, after performing the double-pulse sequence in the magnetic field one makes a light absorption image of the atomic density from which one extracts the number of condensate and noncondensed atoms. Since this imaging technique is sensitive to a specific absorption line of the atoms it does not measure the number of molecules. One also expects to observe

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oscillations in the number of condensate atoms. Moreover, if the situation is such that the detuning between the pulses is relatively large the effect of the coupling can be neglected and the frequency of the observed oscillations corresponds to the energy difference between the atoms and the molecules i.e. the molecular binding energy. This is indeed what is observed, thereby providing compelling evidence for the existence of coherence between atoms and molecules. The parameters of the inter-atomic potentials are fit to the experimental results for the frequency. Clearly, the frequency of the coherent atom-molecule oscillations agrees very well with the molecular binding energy in vacuum over a large range of magnetic field. Moreover in the magnetic field range $B_m \cong 157\text{-}159$ G the frequency of the oscillations is well described by the formulae $[\epsilon_m (B)] = \hbar^2/ma^2 (B)$ for the binding energy. Close to resonance, the measured frequency deviates from the two results [6].

Although some of the physics of these coherent atom-molecule oscillations can be understood by a simple two-level picture. First of all, during Rabi oscillation in a single two level system one quantum in a state oscillates to the other state. In case of Fashbach resonance pairs of atoms oscillate back and forth between the dressed molecular condensate and a atomic condensate. Therefore the Hamiltonian is not quardratic in the annihilation and creation operators and physics is more complicated. In particular the dressed molecule may decay into two noncondensed atoms instead of forming two condensate atoms. Second, the observed atom molecule oscillations are oscillations between an atomic condensate and a dressed molecular condensate[7-10].

II. Materials and Methods

One consider the Hamiltonian of two coupled harmonic oscillators

$$\hat{H} = \frac{1}{2} \left(\hat{a}^\dagger \quad \hat{b}^\dagger \right) \begin{pmatrix} \delta^{(t)} & \delta \\ \Delta & -\delta(t) \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \tag{1}$$

Where $\hat{a}\dagger$ and $\hat{b}\dagger$ create a quantum in the oscillators a and b, respectively, and Δ denotes the coupling between the two oscillators.

One considers first the situation that the detuning δ (t) is time independent. The exact solution is found easily by diagonalizing the Hamiltonian. One assumes that initially there are only quanta in oscillator a and none in b, so that one has that $\langle \hat{b}^\dagger \hat{b} \rangle (0) = 0$. The number of quanta in oscillator a as a function of time is then given by

$$\langle \hat{a}^\dagger \hat{a} \rangle (t) = \left[1 - \frac{\Delta^2}{(\hbar\omega)^2} \sin^2 (\omega t/2) \right] \langle \hat{a}^\dagger \hat{a} \rangle (0) \tag{2}$$

With the frequency ω given by

$$\hbar\omega = \sqrt{\delta^2 - \Delta^2} \tag{3}$$

One sees that the number of quanta in the oscillator a oscillates in time with frequency ω. Such oscillations are called Rabi oscillations. Note that the number of quanta in oscillator b is determined by

$$\langle \hat{b}^\dagger \hat{b} \rangle (t) = - \frac{\Delta^2}{(\hbar\omega)^2} \sin^2 (\omega t/2) \langle \hat{a}^\dagger \hat{a} \rangle (0) \tag{4}$$

So that the total number of quanta is indeed conserved.

Suppose now that one starts from the situation with all quanta in the oscillator a and none is b and that detuning such that δ (t) $>> \Delta$. Then one has from Eq. (2) that $\langle \hat{a}^\dagger \hat{a} \rangle \approx \langle \hat{a}^\dagger \hat{a} \rangle (o)$ and $\langle \hat{b}^\dagger \hat{b} \rangle (t) \simeq (o)$. Starting from this situations, one changes the detuning instantaneously to a value δ (t) $\simeq 0$ and keep it at this value for a time t_{hold}. During this hold time quanta in oscillator a will go to oscillator b. Moreover, t_{hold} is such that

$$t_{hold} \approx \frac{\pi}{2} \frac{\hbar}{\Delta} \tag{5}$$

on average half of the quanta is oscillator a will go to oscillator b. Such a pulse is called a $\pi/2$ pulse. The defining property of a $\pi/2$. pulse is that is creates a superposition of the oscillators a and b, such that the probabilities to be in oscillators a and b equal, and therefore equal to ½. This is indicated by the average $\langle \hat{a}^\dagger \hat{b}^\dagger \rangle$ (t). At t = 0 this average is equal to zero because there is no superposition at that time. One can shows that after the above $\pi/2$- pulse the average $\langle \hat{a} + \hat{b} + \rangle$ (t) reaches its maximum value. In detail, the state after the $\pi/2$ pulse is equal to

$$\frac{1}{\sqrt{N!}} \left[\frac{\hat{a}^\dagger + \hat{b}^\dagger}{\sqrt{2}} \right]^2 | 0 \rangle \tag{6}$$

where the ground state is denoted by $|0\rangle$ and N = $\langle \hat{a}^\dagger \hat{a} \rangle (0)$.

Experimentally, the number of atoms in the condensate is fit to the formulae.

$$N_c(t) = N_{average} - \alpha t + A \exp (-\beta t) \sin (\omega_e t + \phi) \tag{7}$$

Where $N_{average}$ is the average number of condensate atoms, A and ϕ are the oscillation amplitude and phase, respectively, and β is the damping rate of the oscillation. The overall atom loss is characterized by rate constant

α. The experimentally observed frequency is equal to $\omega_e = 2\pi \sqrt{v_e^2 - [\beta/2\pi]^2}$. By defining the frequency of the coherent atom-molecule oscillation in this way one compensates for the effects of the damping on the frequency.

This is seen most easily by considering the linearized version of the time-dependent mean-field equation. Writing

$\phi_a(t) = \phi_a e^{\mu t/\hbar} + \delta\phi_a(t)$ and $\phi_m(t) = \phi_m e^{-2\mu t/\hbar} + \delta\phi_m(t)$, one has that

$$i\hbar \frac{\partial\delta\phi_m(t)}{\partial t} = \left[\delta(B) - g^2 \frac{m^{3/2}}{2\pi\hbar^3} i \sqrt{i\hbar \frac{\partial}{\partial t} - 2\hbar \sum H^F} \right] \delta\phi_m(t) + 2g\phi_a \delta\phi_a(t)$$

$$i\hbar \frac{\partial\delta\phi_a(t)}{\partial t} = 2g\phi_a^* \delta\phi_m(t) \tag{8}$$

where one neglected the off-resonant part of the inter-atomic interactions. This is justified sufficiently close to resonance, where one is also allowed to neglect the energy dependence of the atom molecule coupling constant.

We consider first the situation that the fractional derivative is absent in the linearized meanfield equation in Eq. (8), i.e. one is dealing with the model of Drummond et. al. [11]. and Timmermans et al [12,13]. These coupled equation describe exactly the same Rabi oscillations as the coupled harmonic oscillators in Eq. (1), with the coupling equal to $\Delta = |4g\phi_a|$. In the context of particle-number oscillations between condensates, Rabi oscillations are referred to as Josephson oscillations and the associated frequency is called the Josephson frequency. The Josephson frequency in the absence of the fractional derivative term in Eq. (8) is given by

$$\hbar\omega_J^{bare} = \sqrt{\delta^2(B) + 16g^2 n_a} \tag{9}$$

Which reduces to $\hbar\omega_J^{bare} = |\delta(B)|$ sufficiently for off resonance where the coupling may be neglected. This result does not agree with the experimental result because, by neglecting the fractional derivative, which corresponds to the molecular self-energy, one is describing Josephson oscillations between an atomic condensate and a condensate of bare molecules instead of dressed molecules. Furthermore, using the result in Eq. (2) one has that the amplitude of these oscillation is given by

$$A_J^{bare} = \frac{16g^2 n_a}{[\delta^2(B)]^2} \tag{10}$$

In first approximation one takes the dressing of the molecules into account as follows. If one is in the magnetic field range where the Josephson frequency deviates not too much from the molecular binding energy, one is allowed to expand the propagator of the molecules around the pole at the bound state energy. This corresponds to introducing the dressed molecular field and leads to the Heisenberg equations of motion. The linearized meanfield equations that describe the Josephson oscillations of a atomic and a dressed-molecular condensate are therefore given by

$$i\hbar \frac{\partial\delta\phi_m(t)}{\partial t} = \varepsilon_m(B) \delta\phi_m(t) + 2g\sqrt{Z(B)} \phi_a\delta\phi_a(t)$$

$$i\hbar \frac{\partial\delta\phi_a(t)}{\partial t} = 2g\sqrt{Z(B)} \phi_a^*\delta\phi_m(t) \tag{11}$$

and lead to the Josephson frequency

$$\hbar\omega_j = \sqrt{\varepsilon_m^2(B) + 16g^2 Z(B) n_a} \tag{12}$$

which reduces to $\hbar\omega_j \simeq |\varepsilon_m(B)|$ in the situation where the coupling is much smaller than the binding energy. This result agrees with the experimental fact that the measured frequency is, sufficiently far from resonance, equal to the molecular binding energy. Moreover, the initial deviation from the two-body result in the measured frequency is approximately described by the equation for the Josephson frequency in Eq. (12). The amplitude of the oscillations is in this case given by

$$A_j = \frac{16g^2 Z(B) n_a}{[\varepsilon_m(B)]^2} \tag{13}$$

which close to resonance is much larger than the result in Eq. (10).

With the meanfield theory derived one now calculates the magnetic field and density dependence of the Josephson frequency of the coherent atom-molecule oscillations, in a linear approximation. The only parameter that has not been determined yet is the effective range of the inter-atomic interactions r_{bg}.

The effective range is determined by calculating the molecular binding energy in vacuum and comparing the result with the experimental data. One has seen that for off resonance the Josephson frequency is essentially equal to the molecular binding energy. Since the effect of the nonzero effective range only plays a role for large energies, and thus is important far off resonance, this comparison uniquely determines the effective range. As explained, the molecular binding energy is determined by solving for E in the equation

$$E - \delta(B) - \hbar \sum_m^{(+)} (E) = 0 \tag{14}$$

For ^{85}Rb the background scattering length is negative and the effective range turns out to be positive. The retarded molecular self-energy is therefore given by

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$$\hbar \sum_{m}^{(+)} (E) = \frac{g^2 m}{2\pi \hbar^2 \sqrt{1-2\frac{r_{bg}}{a_{bg}}}} \left[\frac{i\sqrt{1-2(r_{bg}/a_{bg})mE/\hbar^2 - (r_{bg}\,mE/2\hbar^2)}}{1+ia_{bg}\sqrt{1-2(r_{bg}/a_{bg})\,mE/\hbar^2 - (r_{bg}\,a_{bg}\,mE/2\hbar^2)}} \right]$$

(15)

III. Results and Discussion

We have presented the method of evaluation of magnetic field and density dependence of the Josephson frequency of the coherent atom-molecule collisions in the linear approximation. The results are shown in table 1 and 2 respectively by the use of linearized version of the time dependent meanfield equation.

Table- 1: Evaluated result of Josephson frequency of the coherent atom-molecule oscillation as a function of the magnetic field

Magnetic field B(G)	Frequency of the coherent atom-molecule oscillation (Josephson frequency) KHz		
	Our's Cal.	Other's Cal.	Expl.
150	10.2	9.82	12.6
155	15.3	14.68	18.9
156	58.5	55.34	60.6
157	62.6	60.25	65.2
158	69.7	70.56	72.8
159	76.8	76.59	79.5
160	120.5	121.8	122.8
161	460.6	465.6	472.8
162	960.7	970.8	1000.2

Table- 2: Evaluated result of Josephson frequency of the coherent atom-molecule oscillation as a function of the condensate density for fixed magnetic field

n_a (cm^{-3})	Josephson Frequency (KHz)		
	B = 156 G	B = 156.5 G	B = 157 G
1.1×10^{11}	1.25	0.97	0.85
1.5×10^{11}	1.37	1.08	0.86
2.0×10^{11}	1.86	1.18	0.89
3.0×10^{11}	2.58	1.76	1.05
5.0×10^{11}	2.77	2.10	1.25
6.0×10^{12}	3.58	3.86	1.86
8×10^{12}	5.68	4.12	2.65
1×10^{13}	6.72	5.87	3.22
5×10^{13}	7.86	6.10	4.16
8×10^{13}	9.12	8.25	5.75
10×10^{13}	10.68	9.16	6.98
15×10^{13}	12.59	10.25	7.26

This has been developed by Drummond et. al [11]. and Timmermanks et. al. [12,13]. Actually linearized time dependent field equation are coupled equation and these equations describe exactly the Rabi oscillation as the coupled harmonic equation with the coupling constant $\Delta = |4g\phi_a|$. In the context of particle number oscillations between condensates, Rabi oscillations are referred to as Josephson oscillations and the associated frequency is called the Josephson frequency. The Josephson frequency in the absence of fractional derivative is given by

$$\hbar\omega_J^{bare} = \sqrt{\delta^2(B) + 16g^2 n_a}$$
$$\hbar\omega_J^{bare} \cong |\delta^2(B)|$$

n_a is condensate density.

This is Josephson oscillation between and atomic condensate and a condensate of bare molecule instead of dressed molecules. Now if one expand the propagation of the molecules around the pole at the bound state energy then one obtains the Josephson oscillates of an atomic and a dressed molecular condensate and the Josephson frequency is given by

$$\hbar\omega_j = \sqrt{\varepsilon_m^2(B) + 16g^2 Z(B)n_a}$$
$$\cong |\varepsilon_m(B)|$$

This result agrees well with the experimental fact. In table 1 we have given the Josephson frequency $\hbar\omega_j$ in KH$_z$ as a function of magnetic field B(G). We have compared our theoretical result with the work of J.M. Gerton et. al.[14].Our theoretical result are in good agreement with that of the experimental data. In table 2 we have presented the evaluated result of the Josephson frequency of the coherent atom-molecule oscillations as a function of the condensate density n_a for fixed magnetic field. Our theoretical result indicates that that Josephson frequency is almost constant for the condensate density 10^{12}cm^{-3} but it rises very sharply for condensate density

above 10^{13} cm^{-3} as a function of magnetic field B. The magnitude of frequency is low in (KH$_z$) for B= 157 G. Although the trend is same for the given three values of magnetic field B = 156 G, 156.5 G and 157 G. In this calculation we have subtracted the molecular binding energy to bring out the many body effects more clearly. As expected, the difference between the Josephson frequency and the molecular binding energy increases with increasing condensate density. Moreover, for values of the magnetic field closer to resonance the difference is also larger. The above calculations are done in the linear approximation and it gives a great deal of insight in the coherent atom-molecule oscillations and in particular, in their many-body aspects.[15]

IV. Conclusion

From all the theoretical investigation, we get the following results:

(a) In the evaluation of magnetic field and density dependence of the Josephson frequency of the coherent atom molecule, the linear approximation method developed by Drummond et al. and Timmermans et al. works very well in order to reproduce the experimental results.

(b) While over viewing the physics of Josephson effect between Bose Condensed system, it appear that it is possible to explore the crossover between collective Josephson behavior and independent boson Rabi dynamics.

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