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Transition to Coarse-Grained Order in Coupled Logistic Maps: Effect of Feedback

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Abstract: *The work reported here is an extension to the previous workby the authors on coupled logistic maps with delayed linear, or non-linear nearest neighbor coupling. Here we investigate the effect of introduction of feedback in the coupled lattice update rules on (i) the phase diagrams showing non-zero persistencein the $\mu - \varepsilon$ parameter space, and (ii) on power law exponents of decay of persistence. We find that while feedback expands the non-zero persistence regions in the phase diagram, it has no discernible effect on the power law exponents. We also find that the role played by time lagin the absence of feedback, discussed in our previous work,remains intact even with the introduction of feedback.*

Keywords:*delay, feedback, logistic map, persistence, universality,*

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I. Introduction

Phase transitions, especially second order phase transitions, have been studied for many years now. Second order phase transitionsin equilibrium thermodynamic systems do not show coexistence of phases [1]and also do not involve any latent heat. Magnetic transitions are the best example of second order phase transitions. The transition can be said to be associated with emergence of a global order described by an order parameter – the net magnetization of the phase.Studies in phase transitions have considered non-equilibrium transitions and focused on search for an order parameter, which would be non-zero in the emerging phase. The associated power laws and universalityclass [2] have been of concern in these studies.

An important class of models of non-equilibrium systems is coupled map lattice (CML) models. This class of models is computationally more efficient and economical compared the other models. CML[3] models have been extensively studied for the past few decades. These are basic models for time evolution of systems with non-linear interactions. They are discrete space, discrete time models, in which real values, or vectors are assigned to a lattice of points in space. Time evolution of the system is defined by dynamical equations of the model which are in the form of update rule for site vectors.

The present work is focused on phase transitions in one dimensional CML models with logistic map, defined by$f(x) = \mu x\,(1 - x)$as local dynamics. Such models have been used for long for non-equilibrium phase transitions. The models studied here incorporate either linear, or non-linear nearest neighbor (NN) coupling and feedback with time lag. The logistic map parameter,strength of delayed NN-coupling and feedback,and the value of time delay, (or time lag) provide parameters of the CML. We begin by randomly initializingthe lattice site values and evolve them using dynamical equations defining the CML model, examining them after each time step to assign a spin value by the following rule:Up spin for those values which are above or equal to the fixed point $x^*$of the logistic map and down spin to those below it. The fixed point is given by the equation $x^* = 1 - 1/\mu$. The sites, which retain their original spin even after certain number of time steps,are said to be persistent sites. Fraction of persistent sites at any time step defines the persistence probability at that time. We look for CML parameter sets for which asymptotic persistence probability is non-zero. For fixed values of time lag and feedback strength, non-zero persistence providespairs of values of the remaining parameters – logistic parameter and coupling strength –making up a phase diagram for the system.We investigate existence of a power law for persistence in time for CML models on a linear partof boundary of the phase diagramlying within the region $0 \leq \varepsilon \leq 0.17$.

The CML models we investigate have the following dynamics for updating site values:

$$x_i(t+1) = \left[(1-\beta-\varepsilon)f\big(x_i(t)\big) + \frac{\varepsilon}{2}\big(x_{i-1}(t-\tau) + x_{i+1}(t-\tau)\big) + \beta x_i(t-\tau)\right] (\text{mod } 1) \quad (1a)$$

for linear feedback and NN Coupling, and

$$x_i(t+1) = \left[(1-\beta-\varepsilon)f\big(x_i(t)\big) + \frac{\varepsilon}{2}(f(x_{i-1}(t-\tau)) + f(x_{i+1}(t-\tau))) + \beta f(x_i(t-\tau))\right] (\text{mod } 1) \quad (1b)$$

For the non-linear case. Thus, the non-linearity in feedback and coupling is defined by the same function f, which in our case is the logistic map. Also, the time lag in both the coupling and the feedback is taken to be the same. Here, β and ε are respectively the feedback and the NN-coupling strengths and τ, the delay, or time lag. The index i ranges over 1 to N, the total number of lattice sites, or the lattice size. As is usual, we impose cyclic boundary conditions, where the last lattice point is the neighbor of the first. The variables $x_i(t) \in [0,1]$ are the real values attached at time-step t to the i^{th} lattice point of a one-dimensional lattice of size N.

For small values of ϵand β, the dominant part of dynamics is given by the logistic map. It is known that for much of the unit interval the map sends a value $x < x^*$ ($x > x^*$) to another $> x^*$ (respectively, $< x^*$), i.e., it sends up (down) spin to down (respectively, up) spin. Since we are looking for local spin-persistence, it is natural to define a single time-step in our CML dynamics by a double application of the update rules above. We call this modulo-2 dynamics.

II. The Plots

We present the numerical results in three parts: phase plots, persistence plots and long-range order plots.

2.1 Phase Plots

In this section we present results of numerical computation with several specific cases of CML dynamics. The non-linear case in Eq(1b) with $\beta = 0.0$ and $\tau = 0$, i.e., without feedback and delay has been extensively studied by Gade and Sahasrabudhe [4], Here, we extend the study to cases with non-zero feedback and delay values as well as those with linear NN coupling. (Fig.1) below shows the phase plots in a part of the $\mu - \varepsilon$ plane defined by $3.6 \leq \mu \leq 4; 0.0 \leq \epsilon \leq 1.0$. Computed for the CMLs of Eq(1a), (1b) with feedback and coupling with delays $\tau = 0, 1, 2$, they show regions of non-zero asymptotic persistence. The zero persistence regions are white. We note that a large part of the $\mu - \varepsilon$ region shown allows non-zero persistence. Moreover, in case of non-linear coupling delay seems to have little effect on the phase plots, whereas, odd delays seem to wipe out a considerable part of the non-zero persistence region when the coupling is linear. Lastly, we note a more, or less well-defined line in the $3.65 < \mu \leq 4, 0.0 \leq \varepsilon < 0.17$ region in all the phase plots separating zero and non-zero persistence.

Each case, whether it is of linear or nonlinear coupling is iterated for 10^4 sites over10^6, or more time steps to confirm the power law. The lower portions of the μ-ε phase plots define a critical line separating the zero and the nonzero persistence. Although these plots show variationwith feedback and time lag values above the critical line, the line itselfmore, or less persists. The phase plots for linearly coupled CML fill the plot region more extensively than the non-linearly coupled one.However, this does not seem to affect the critical linemuch for very small values of feedback strength. This changes for higher values of feedback. We see a rightward shift of the critical line along the μ-axis -the higher the feedback, the larger the shift and the smaller the μ-range over which it extends.

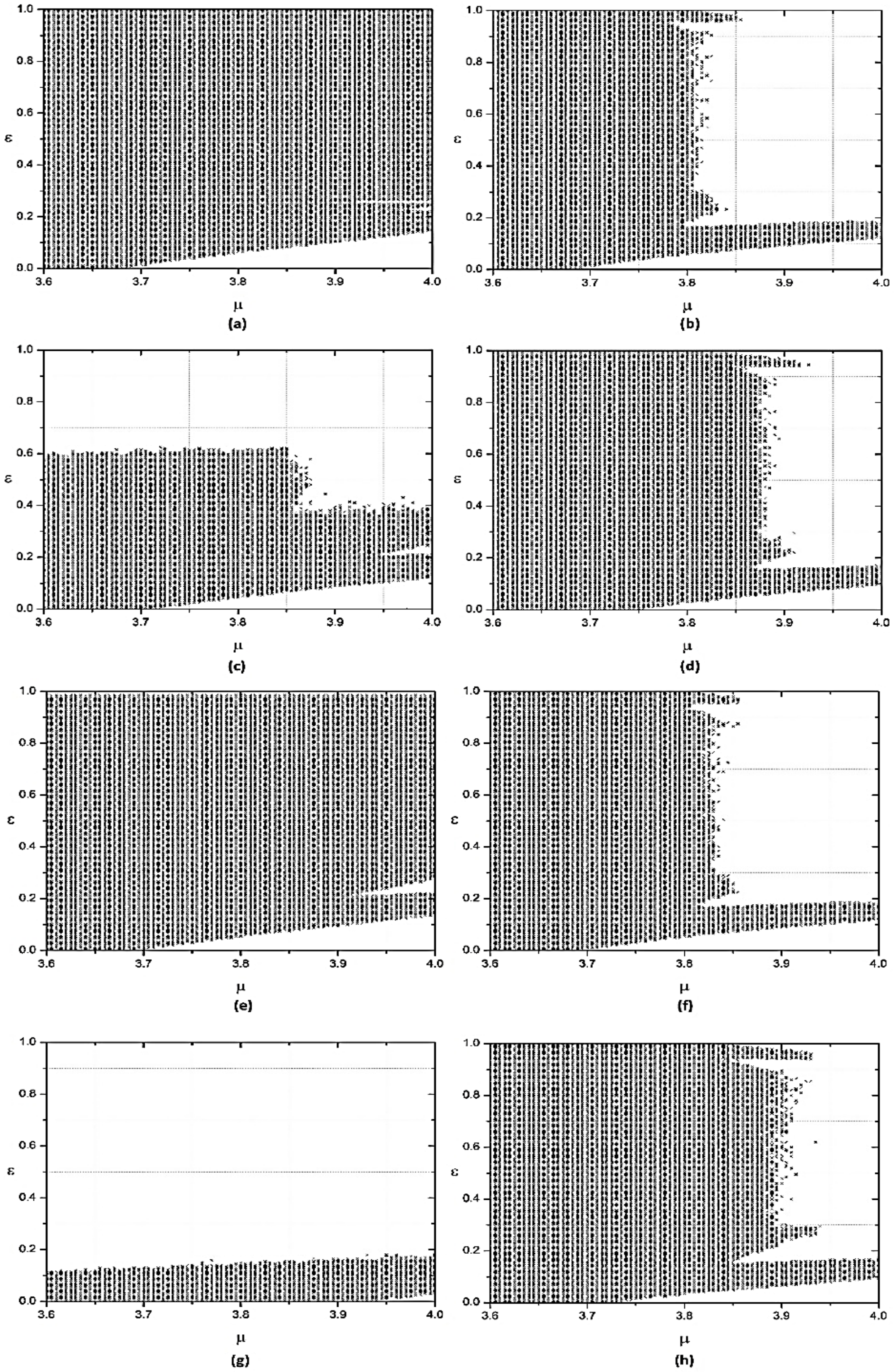

Fig.1: Phase plots for both linear and non-linear coupling cases with feedback: **(a)** linear,$\tau = 0,\ \beta = 0.025$,**(b)** nonlinear, $\tau = 0,\ \beta = 0.025$,**(c)** linear, $\tau = 1,\ \beta = 0.0125$, **(d)** nonlinear, $\tau = 1,\ \beta = 0.0125$, **(e)** linear, $\tau = 2, \beta = 0.0125$, **(f)** nonlinear, $\tau = 0,\ \beta = 0.0125$,**(g)** linear, $\tau = 3, \beta = 0.085$, **(h)** nonlinear, $\tau = 3,\ \beta = 0.025$. Apart from the features mentioned in the text for zero feedback case, for non-zero delays here we notice a certain apparently β-dependent right-ward shift in the critical line. For both linear and non-linearcouplingthe shift is larger for odd delays.

2.2 Persistence Plots

We refer to the points on the critical line(the lower boundary separating zero and non-zero persistence regions)as critical points – each point being a μ-ε pair of values. These pairs are then used in a persistence probability P(t) calculation over 10^6, or more time steps to generate a log-log plots of persistence points vs. time in order to ascertain a power lawThe power law$\mathrm{P(t)} \propto \mathrm{t}^{-\theta}$, leading to a linear log-log plot, defines persistence exponentθ[5].As noted above, feedback leads to a rightward shift of the critical line shortening the μ-range over which it extends.We find that this effect of feedback notwithstanding, the log-log plots show the same exponents as have been reported in earlier work by the same authors [5, 6] with zero feedback and zero initialization of time-lagged lattice values. The work here also shows that random initialization of time-lagged lattice valuesdoes not change the delay dependent exponents, or even the emergent long-range order(**2.3** below) of the system. Our computation shows that odd and even delayslead to two different values ofθ.(Fig.2) shows the results for linear NN-coupling.

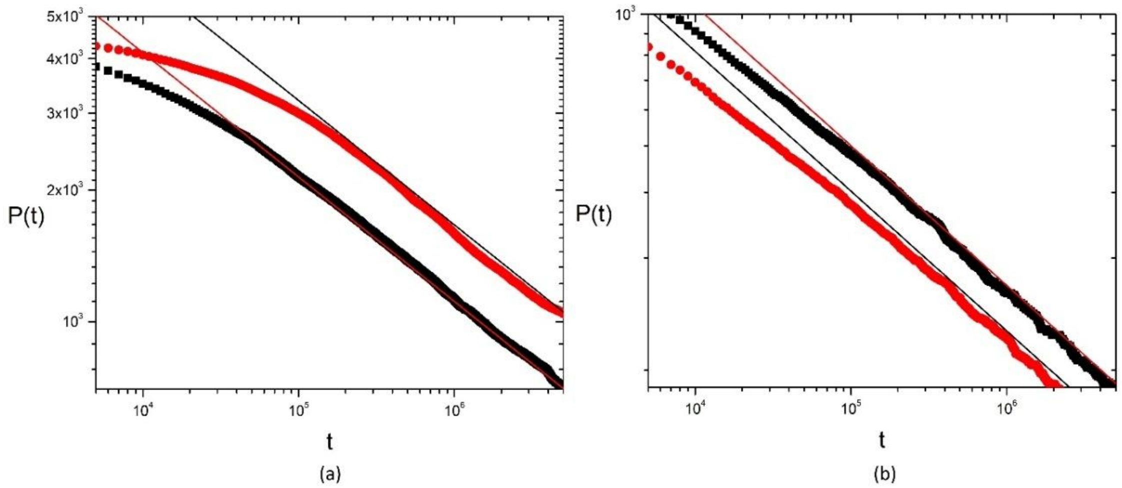

Fig2: Persistence plots for linear and NN coupling with same feedback of$\beta = 0.025$cases as:**(a)** even delays$\tau = 0, (\mu, \varepsilon) = (3.9, 0.1)$ (black) and $\tau = 2, (\mu, \varepsilon) = (3.9, 0.11)$ (red)showing exponent$\mathrm{P(t)} \approx \mathrm{t}^{2/7}$and**(b)** odd delays$\tau = 1, (\mu, \varepsilon) = (3.9, 0.109)$(black) and $\tau = 3, (\mu, \varepsilon) = (3.9, 0.11)$(red)showing exponent $\mathrm{P(t)} \approx \mathrm{t}^{3/8}$.

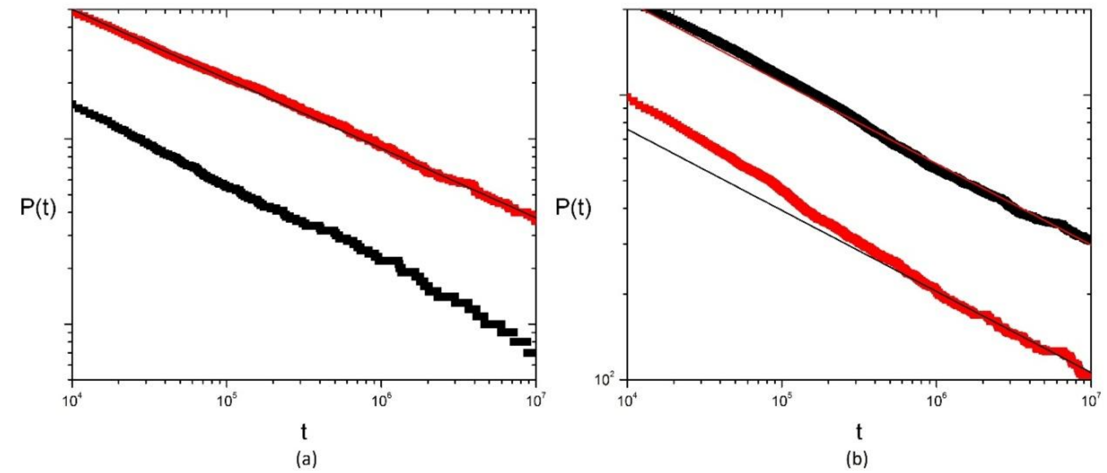

Fig.3: Persistence plots for nonlinear NN coupling with same feedback of$\beta = 0.025$cases as: **(a)**evendelays as $\tau = 0, (\mu, \varepsilon) = (3.9, 0.12)$(black), $\tau = 2, (\mu, \varepsilon) = (3.9, 0.1)$(red) showing exponent $P(t) \approx t^{3/8}$ and**(b)** odd delays$\tau = 1, (\mu, \varepsilon) = (3.9, 0.095)$ (black), $\tau = 3, (\mu, \varepsilon) = (3.9, 0.091)$ (red) with exponent$P(t) \approx t^{2/7}$.

(Fig.3) shows the same with nonlinear NN coupling. We find that change in nature of coupling causes an exchange of exponents between even and odd delay cases. The exponent2/7,obtained for even delays now gets associated with the odd delays, whereas exponent 3/8 is now obtained for even delays. Also, to ascertain whether these assertions hold for stronger feedbacks too, persistence computations were onefor non-zero delays and increasingfeedback strengthsβranging up to values limited only by the eventual disappearance of critical line in the phase plots (see Fig.1) signifying spread of persistence region right up to zero NN-coupling (μ-axis).

The persistence plots (Fig.4) show no change in the exponents. The linear NN coupling case showsthe same exponent of 2/7, while the nonlinear coupling case shows3/8. Thus, persistence exponent is governed onlyby the type of coupling and the value of delay, with no dependence on feedback strength.

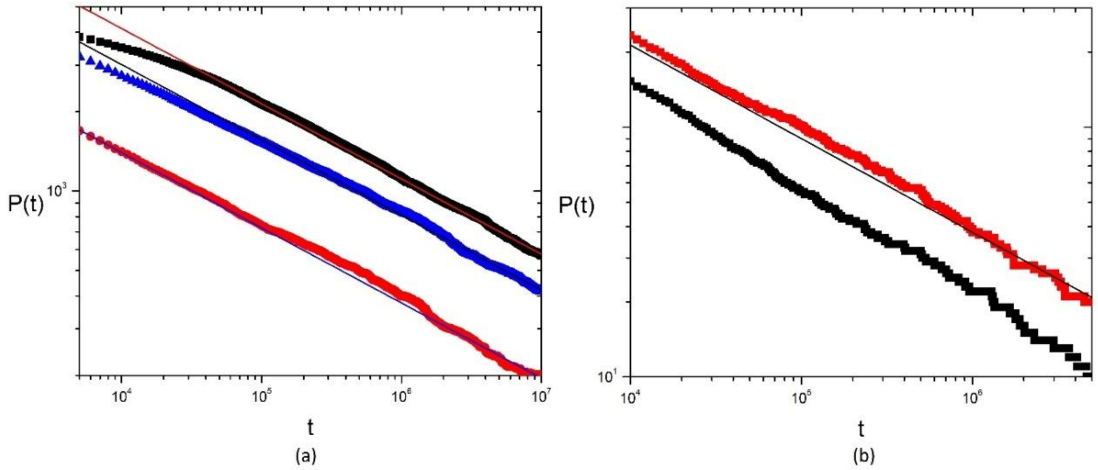

Fig4:Persistence plots with delays$\tau = 0$ cases as:**(a)**for feedbacks of$\beta = 0.025$,$(\mu, \varepsilon) = (3.9, 0.1)$(black),$\beta = 0.05, (\mu, \varepsilon) = (3.9, 0.045)$(red) and $\beta = 0.075, (\mu, \varepsilon) = (3.9, 0.003)$(blue) for linear NN coupling showing exponent of $P(t) \approx t^{2/7}$ and **(b)** with feedbacks of$\beta = 0.025$, $(\mu, \varepsilon) = (3.9, 0.12)$(black), and $\beta = 0.05, (\mu, \varepsilon) = (3.9, 0.1)$(red) for nonlinear NN coupling showing the exponent$P(t) \approx t^{3/8}$.

2.3 Long Range Order Plot

Emergence of asymptotic long-range order in the CML was examined to assert its association with the persistence exponent.For linearly coupled CML the resulting plots show that even delay with persistence exponent of 2/7 leads to ferromagnetic order. On the other hand, for a non-linearly coupled CML even delay and persistence exponent of 3/8 produces antiferromagnetic order. This result is replicates that reported earlier by us for the case of zero feedback. Moreover, the important role played by the time lag in zero feedback case is not affected by introduction of feedback. For the linear case, the ferromagnetic order obtained with $\tau = 0$ switches to antiferromagnetic order for $\tau = 1$. On the other hand, in the non-linearly coupled system, the order switches from an antiferromagnetic one for $\tau = 0$ to a ferromagnetic one for $\tau = 1$. For each unit increase in τ this switch is repeated in both types of coupling cases. Thus, in the linear case, the long-range order settles to a ferromagnetic one for even τ values, and to an antiferromagnetic one for odd τ values. In the non-linear case, the roles of these two orders are interchanged. Thus, as far as asymptotic long-range order is concerned, we find the following reciprocal relationship between linear and non-linear coupling: For odd (even) values of τ, a linearly coupled lattice behaves like a non-linearly coupled one with even (respectively, odd) values of τ, and conversely.

(Fig.5) shows this reciprocal relationship by bringing together the plots showing emergence of long-range order for different delay values for the two types of coupling into separate columns and those for both the types of coupling for the same delay value into different rows. Each plot shows spin-up sites in black at successive time steps. Emergence of ferromagnetic (antiferromagnetic) order is signaled by parallel black lines for all (respectively alternate) site indices in the upper part of each plot in the figure.

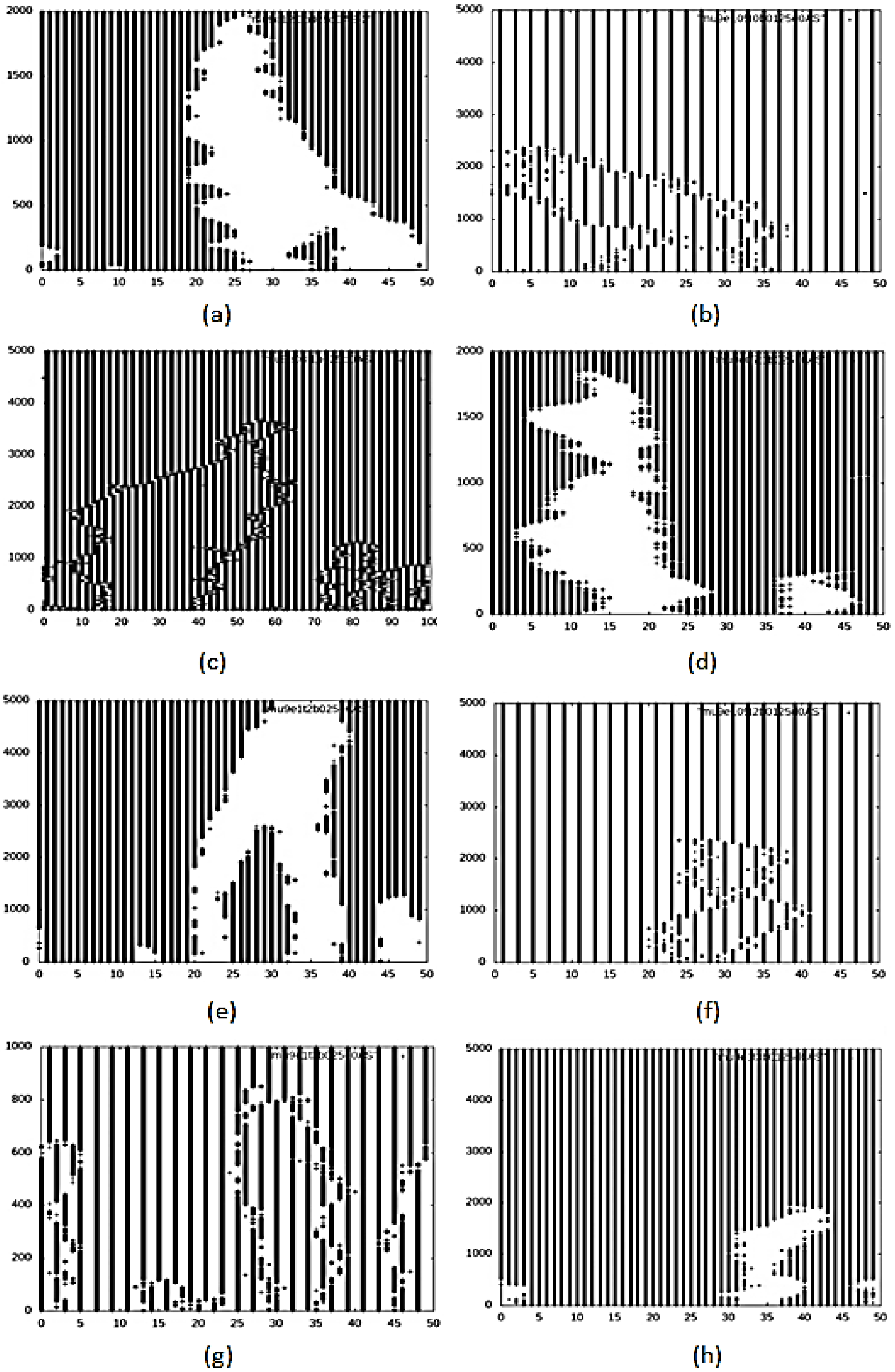

Fig.5: Emergence of long-range order inCML with linear and nonlinear NN-coupling: **(a)**linear,$\tau = 0, \beta = 0.025, (\mu, \varepsilon) = (3.9, 0.12)$, **(b)**nonlinear,$\tau = 0, \beta = 0.0125, (\mu, \varepsilon) = (3.9, 0.105)$,**(c)**linear,$\tau = 1, \beta = 0.025, (\mu, \varepsilon) = (3.9, 0.09)$,**(d)**nonlinear,$\tau = 1, \beta = 0.025, (\mu, \varepsilon) = (3.9, 0.07)$, **(e)** linear,$\tau = 2, \beta = 0.025, (\mu, \varepsilon) = (3.9, 0.1)$**(f)** nonlinear,$\tau = 2, \beta = 0.0125, (\mu, \varepsilon) = (3.9, 0.105)$, **(g)**linear,$\tau = 3, \beta = 0.025, (\mu, \varepsilon) = (3.9, 0.1)$**(h)** nonlinear,$\tau = 3, \beta = 0.0125, (\mu, \varepsilon) = (3.9, 0.1)$. The rightward axis shows site indices, and the upward axis shows time steps. In **(c)** the number of sites is 100, and only alternate ones are spin-up (black) implying an antiferromagnetic order.

III. Conclusion

The main conclusions of the work on linearly, or non-linearly coupled logistic map lattices reported here are**(i)**that introduction of feedback leads to increase in the incidence of asymptotic persistence, thus adding regions to phase diagrams in the $\mu - \varepsilon$ plane, **(ii)**that feedback has no effect on the power law exponents,honoring even their dependence on time lag, which was discovered in our earlier work [5, 6], and **(iii)** that it does not disturb the emergence of long range order, which one obtains in the zero feedback case[6].Persistence exponentsand long-range order (ferromagnetic, or antiferromagnetic) are thus determined only by the type (linear, or non-linear) of NN-coupling and by parity (even /odd nature) of time lag; neither of these is affected by feedback.

We have earlier interpreted the effect of time lag and reciprocity between even-valued (odd-valued) lags in linear coupling and odd-valued (respectively, even-valued) lags in non-linear coupling in terms of a "spin-flipping fraction" identifiable at each modulo-2 dynamical step in evolution of the lattice endowed with coarse-grained association of spin with lattice site values. The spin-flipping fraction, which we may identify with $P(t) - P(t+1)$ at time t, is essentially the fraction of site values crossing x* in the t'th modulo-2 dynamical update of the lattice. It is responsible for the power-law decay in persistence. It is also responsible for the emergence of long-range order to either a ferromagnetic, or antiferromagnetic one. It is known that the Ising model at zero temperature displays ferromagnetic order when the spin-spin coupling is positive (J>0), and an antiferromagnetic order when the coupling is negative (J<0). Earlier, we have argued that the CML models studied have features displayed by Glauber dynamics on Ising model [7] with unequal probabilities of up-to-down and down-to-up spin-flips[4]. Due to the nature of logistic map definition of dynamics as a double application of update rules of Eq(1) at each time step, a unit delay in coupling in the CML studied here affects the spin-flipping fraction in such a manner as toconvert a positive coupling to a negative. This switches the long-range order emerging asymptotically from ferromagnetic to antiferromagnetic. The switch is repeated with each unit increase in delay. A major feature of dynamics thrown up by the present work is that feedback has no such effect on the spin-flipping fraction, thus leaving both persistence exponent and emergent long-range order unaffected, and otherwise determined only by coupling type and parity of time lag. However, feedback does affect theincidence of persistence itself, bringing larger regions of phase space under the critical line into the phase diagram.

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