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Notes on Fidelity of Coherence

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Abstract: *As a coherent measure in quantum theory resources, the coherence measure included by fidelity is discussed in this paper. We first study the fidely coherence of the maximum coherent state ρ in C^d , and then find that its optimal incoherence states is ρ_{diag} , finally, using the YZX framework of coherence measure we prove that the fidelity of coherence is not a reasonable coherence measure.*

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I. Introduction

As the basic feature of quantum mechanics, quantum coherence is an important part of quantum information processing [1], and has many important applications in various disciplines, such as quantum optics [2], quantum computation [3-5], quantum biology [6,7], and thermodynamics [8-10].

Similar to the established method of entanglenent resources [11,12], T. Baumgratz, M. Cramer, and M. B. Plenio [13] first proposed a quantitative coherence framework including four conditions, which is called BCP framework. Since then, a series of coherence measures are proposed based on this framework, such as the relative entropy of coherence, the l_1 norm of coherence, and the formation of coherence [13-16]. Although the BCP framework is widely used as a way to measure coherence, there is still controversy about its framework. In later studies, Xiao-Dong Yu, Da-Jian Zhang, G. F. Xu, and D. M. Tong [17] proposed an equivalent framework including three basic conditions, which is called YZX framework. The above conditions in YZX framework applied to a variety of physical environments and compatible with the BCP framework. By using the YZX framework, many open questions and arguments can be easily resolved. Using YZX framework, it was easily to prove [17] that the trace norm of coherence is not a legitimate coherence but the modified trace norm of coherence is a legitimate coherence.

Fidelity is the degree to which the receiver output signal is close to the modulated signal waveform. For example, a j-line electric receiver or a television set, the more realistic the sound or image output, that is, the smaller the distortion, the higher the fidelity [18]. Therefore, it is very meaningful to study fidelity. And it has been prove in [19] that the fidelity of coherence does not satisfy strong monotonicity in BCP framework. Then a natural problem arose that whether the fidelity of coherence satisfies the conditions of the YZX framework. In this paper, we first study the fidelity of the largest coherent state and find its closest incoherent state under fidelity coherence measure, then using YZX framework we prove that the fidelity of coherence is not a legitimate coherent measure.

II. Preliminaries

In this section, we first recall two quantitative coherence frameworks. In 2014, Baumgratz et al. [13] proposed the following BCP framework for quantifying coherence as a resource.

Definition 1. [13] A functional C can be taken as a coherence measure if it satisfies the following four conditions:

(C1) $C(\rho) \geq 0$ for all states, and $C(\rho) = 0$ if and only if ρ is an incoherence states;

(C2a) $C(\rho) \geq C(\Lambda(\rho))$ if Λ is an incoherence operation, i.e., a completely positive trace-preserying (CPTP) map $\Lambda(\rho) = \sum_n K_n \rho K_n^+$ with the Kraus operators K_n satisfying $K_n I \ K_n^+ \subset I$, where I is the set of incoherence states;

(C2b) $C(\rho) \geq \sum_n p_n C(\rho_n)$, where $p_n = Tr\left(K_n \rho K_n^+\right)$, $\rho_n = K_n \rho K_n^+ / p_n$, and K_n are the Kraus

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operators of an incoherent CPTP map $\Lambda(\rho)=\sum_n K_n \rho K_n^+$;

(C3) $\sum_n p_n C(\rho_n) \geq C\left(\sum_n p_n \rho_n\right)$ for any set of states $\{\rho_n\}$ and any probability distribution $\{p_n\}$.

In 2016, Xiao-Dong Yu et al. [17] proposed YZX framework for quantifying coherence, which is equivalent to the BCP framework.

Definition 2. [17] A functional C can be taken as a coherence measure if it satisfies the following three conditions:

(B1) $C(\rho) \geq 0$ for all states, and $C(\rho)=0$ if and only if ρ is an incoherent state;

(B2) $C(\rho) \geq C(\Lambda(\rho))$, if Λ is an incoherent operation;

(B3) $C(p_1\rho_1 \oplus p_2\rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$ for block-diagonal states $p_1\rho_1 \oplus p_2\rho_2$ in the incoherent basis.

Obviously, (C1) and (B1) are the same, (B2) and (C2a) are the same, moreover it has been proved that the condition (B3) is equivalent to the conditions (C2b) and (C3) [17].

III. The Fidelity of Coherence

In 2014, Lian-He Shao et al. [19] have proved that the fidelity of coherence is not a legitimate coherence measure using BCP framework. In this section, we use YZX framework to verify that the fidelity of coherence is not a reasonable coherence measure.

In 2014, the fidelity of coherence was defined as follows:

$$C_F(\rho) = \min_{\delta \in I} D(\rho,\delta) = 1 - \sqrt{\max_{\delta \in I} F(\rho,\delta)}. \qquad (1)$$

where $F(\rho,\delta) = \left[tr\sqrt{\rho^{\frac{1}{2}}\delta\rho^{\frac{1}{2}}} \right]^2$, which is non-decreasing under CPTP maps Λ, e.g., $F(\Lambda(\rho),\Lambda(\delta)) \geq F(\rho,\delta)$ [20]. It has been proved in [13] that the fidelity of coherence (1) fulfills $(\mathrm{C1}),(\mathrm{C2a}),(\mathrm{C3})$, and it does not satisfy (C2b) in BCP framework [19].

Lemma 1. For the maximum coherent state in C^d, $\rho=|\psi_d\rangle\langle\psi_d|$, where $|\psi_\mathrm{d}\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle$, its fidelity of coherence has the following form:

$$C_F(\rho) = C_F(|\psi_d\rangle\langle\psi_d|) = 1 - tr\sqrt{\frac{1}{d}|\psi_d\rangle\langle\psi_d|} = 1 - tr\sqrt{\frac{1}{d}\rho}. \qquad (2)$$

Proof. Let

$$U_n = \sum_{k=0}^{d-1} |k \oplus_d n\rangle\langle k|, \quad n = 0, 1, 2, \ldots, d-1.$$

then we have

$$\begin{aligned} U_n|\psi_d\rangle &= \sum_{k=0}^{d-1}|k\oplus_d n\rangle\langle k|\frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle \\ &= \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\oplus_d n\rangle \\ &= \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle \\ &= |\psi_d\rangle, \end{aligned} \qquad (3)$$

and

$$\sum_{n=0}^{d-1} U_n \delta U_n^+ = \sum_{n=0}^{d-1}\sum_{k=0}^{d-1}|k\oplus_d n\rangle\langle k|\sum_{i=0}^{d-1}\delta_{ii}|i\rangle\langle i|\sum_{k'=0}^{d-1}|k'\rangle\langle k'\oplus_d n|$$

$$
\begin{aligned}
&= \sum_{n=0}^{d-1}\sum_{k=0}^{d-1}\delta_{kk}\left|k\oplus_d n\right\rangle\left\langle k\oplus_d n\right| \\
&= \sum_{k=0}^{d-1}\delta_{kk}\sum_{n=0}^{d-1}\left|k\oplus_d n\right\rangle\left\langle k\oplus_d n\right| \qquad (4)\\
&= \sum_{k=0}^{d-1}\left(\delta_{kk}+\delta_{k\oplus_d 1,k\oplus_d 1}+\cdots+\delta_{k\oplus_d (d-1),k\oplus_d (d-1)}\right)\left|k\right\rangle\left\langle k\right| \\
&= \sum_{k=0}^{d-1}\left|k\right\rangle\left\langle k\right| = I_d .
\end{aligned}
$$

By unitary invariance of fidelity, $F\left(U\rho U^{+},U\delta U^{+}\right)=F\left(\rho,\delta\right)$, we get that

$$
\begin{aligned}
F\left(\rho,\delta\right) &= F\left(\left|\psi_d\right\rangle\left\langle\psi_d\right|,\delta\right) \\
&= F\left(U\left|\psi_d\right\rangle\left\langle\psi_d\right|U^{+},U\delta U^{+}\right) \qquad (5)\\
&= \frac{1}{d}\sum_{n=0}^{d-1}F\left(U_n\left|\psi_d\right\rangle\left\langle\psi_d\right|U_n^{+},U_n\delta U_n^{+}\right).
\end{aligned}
$$

then by the concavity of fidelity, we have

$$
F\left(\sum_i p_i\rho_i,\sum_i p_i\delta_i\right)\geq\sum_i p_i F\left(\rho_i,\delta_i\right). \qquad (6)
$$

then from the above relationship (5) and (6), we have

$$
\begin{aligned}
F\left(\rho,\delta\right) &= F\left(\left|\psi_d\right\rangle\left\langle\psi_d\right|,\delta\right) \\
&= \frac{1}{d}\sum_{n=0}^{d-1}F\left(U_n\left|\psi_d\right\rangle\left\langle\psi_d\right|U_n^{+},U_n\delta U_n^{+}\right) \qquad (7)\\
&\leq F\left(\frac{1}{d}\sum_{n=0}^{d-1}U_n\left|\psi_d\right\rangle\left\langle\psi_d\right|U_n^{+},\frac{1}{d}\sum_{n=0}^{d-1}U_n\delta U_n^{+}\right) \\
&= F\left(\left|\psi_d\right\rangle\left\langle\psi_d\right|,\frac{1}{d}I_d\right).
\end{aligned}
$$

hence

$$
\begin{aligned}
C_F\left(\rho\right) &= 1-\sqrt{\max_{\delta\in I}F\left(\rho,\delta\right)} \\
&= 1-\sqrt{F\left(\left|\psi_d\right\rangle\left\langle\psi_d\right|,\frac{1}{d}I_d\right)} \qquad (8)\\
&= 1-\sqrt{\left[tr\sqrt{\left(\left|\psi_d\right\rangle\left\langle\psi_d\right|\right)^{\frac{1}{2}}\frac{1}{d}I_d\left(\left|\psi_d\right\rangle\left\langle\psi_d\right|\right)^{\frac{1}{2}}}\right]^2} \\
&= 1-tr\sqrt{\frac{1}{d}\left|\psi_d\right\rangle\left\langle\psi_d\right|}=1-tr\sqrt{\frac{1}{d}\rho} \quad .
\end{aligned}
$$

Lemma 2. Under the fidelity of coherence (1), the closest incoherent state to the largest coherent state $\rho=\left|\psi_d\right\rangle\left\langle\psi_d\right|$ in C^d is $\rho_{diag}=\frac{1}{d}I_d$.

Proof. It follows from (7) in the proof of Theorem 1.

Next, we will verify that the fidelity of coherence (1) does not satisfy the condition (B3) of YZX framework.

Theorem 1. The fidelity of coherence is not a legitimate coherence measure.

Proof. Consider the following state in C^5,

$$\rho = \frac{1}{2}\rho_1 \oplus \frac{1}{2}\rho_2 .$$

with $\rho_1 = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|),\ \rho_2 = \frac{1}{3}(|2\rangle + |3\rangle + |4\rangle)(\langle 2| + \langle 3| + \langle 4|).$

From (7), we have

$$C_F(\rho_1) = 1 - tr\sqrt{\frac{1}{2}\rho_1} = 1 - \frac{1}{\sqrt{2}} .$$

$$C_F(\rho_2) = 1 - tr\sqrt{\frac{1}{3}\rho_2} = 1 - \frac{1}{\sqrt{3}} .$$

So

$$\frac{1}{2}C_F(\rho_1) + \frac{1}{2}C_F(\rho_2) = 1 - \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{3}} , \qquad (9)$$

moreover,

$$C_F(\rho) = 1 - \sqrt{\max_{\delta \in I} F(\rho, \delta)} \le 1 - \left(tr\sqrt{\rho^{\frac{1}{2}}\delta_0\rho^{\frac{1}{2}}}\right)^2 = 1 - \frac{1}{\sqrt{6}} - \frac{1}{3\sqrt{2}} . \qquad (10)$$

where $\delta_0 = diag\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right).$

Comparing the results of (9) and (10), we can get

$$\frac{1}{2}C_F(\rho_1) + \frac{1}{2}C_F(\rho_2) > 1 - \sqrt{F(\rho, \delta_0)} \ge C_F(\rho).$$

hence, $C_F(\rho) \ne \frac{1}{2}C_F(\rho_1) + \frac{1}{2}C_F(\rho_2).$

Thus, we have proved that the fidelity of coherence does not satisfy the condition (B3), so according YZX framework, the fidelity of coherence is not a legitimate coherence measure.

IV. Conclusion

In this paper, we discussed the form of fidelity coherence of the maximum coherent state ρ, and found that its closest incoherent state is $\rho_{diag} = \frac{1}{d} I_d$. Moreover, based on the YZX framework, we proved that the fidelity of coherence does not satisfy the condition (B3). We then conclude that the measure of coherence induced by fidelity is not a good measure for quantifying coherence.

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