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The Golden Mean

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Abstract: *This paper is all about golden ratio Phi =*
1.618033988749894848204586834365638117720309179805762862135448622705260462818902
2449707207204189391137484754088075386891752126633862223536931793180060766726354
4333890865959395829056383226613199282902678806752087668925017116962070322210
4321626954862629631361443814975870122034080588795445474924618569536486444924104
432077134494704956584678850987433944221254487706647809158846074998871240076521
705751797883416625624940758906970400028121042762177111777805315317141011704666
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909501680328112194320481964387675863314798571911397815397807476150772211750826
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683437670235037111633072586988325871033632223810980901211019899176841491751233
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297756378925999164994642819302229355234667475932695165421402109136301819472270
789012208728736170734864999815625547281137347987165695274890081443840532748378
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627701243679831144643694767053127249241047167001382478312865650649343418039 ... Φ

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I. Introduction

The concept of golden ratio division appeared more than 2400 years ago as evidenced in art & architecture. It is possible that the magical golden ratio divisions of parts are rather closely associated with the notion of beauty in pleasing, harmonious proportions expressed in different areas of knowledge by scientists, biologists, physicist, artists, musicians, historians, architects, psychologists, and even mystics. For example, the Greek sculptor Phidias (490–430 BC) made the Parthenon statues in a way that seems to embody the golden ratio; Plato (427–347BC), in his Timaeus, describes the five possible regular solids, known as the Platonic solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron), some of which are related to the golden ratio. The properties of the golden ratio were mentioned in the works of the ancient Greeks Pythagoras (c. 580–c. 500 BC) and Euclid (c. 325–c. 265 BC), the Italian mathematician Leonardo of Pisa (1170s or 1180s–1250), and the Renaissance astronomer J. Kepler (1571–1630). Specifically, in book VI of the*Elements,* Euclid gave the following definition of the golden ratio: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less". Therein Euclid showed that the “mean and extreme ratio", the name used for the golden ratio until about the 18^{th}century, is an irrational number. In 1509 L. Pacioli published the book*Divine Proportion,* which gave new impetus to the theory of the golden ratio; in particular, he illustrated the golden ratio as applied to human faces by artists, architects, scientists, and mystics. G. Cardano (1545) mentioned the golden ratio in his famous book *Ars Magna,* where he solved quadratic and cubic equations and was the first to explicitly make calculations with complex numbers. Later M. Mästlin (1597) the reciprocal. J.Kepler (1608) showed that the ratios of Fibonacci numbers approximate the value of the golden ratio and described the golden ratio as a "precious jewel”. Throughout history many people have tried to attribute some kind of magic or cult meaning as a valid description of nature and attempted to prove that the golden ratio was incorporated into different architecture and art objects (like the Great Pyramid, the Parthenon, old buildings, sculptures and pictures). But modern investigations (for example, G. Markowsky (1992), C. Falbo (2005), and A. Olariu(2007) showed that these are mostly misconceptions: the differences between the golden ratio and real ratios of these objects in many cases reach 20–30% or more.

Some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. Biologists, artists, musicians, historians, architects, psychologists, and even mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.

Ancient Greek mathematicians first studied what we now call the golden ratio because of its frequent appearance in geometry; the division of a line into "extreme and mean ratio" (the golden section) is important in the geometry of regular pentagrams and pentagons. According to one story, 5th-century BC mathematician Hippasus discovered that the golden ratio was neither a whole number nor a fraction (an irrational number), surprising Pythagoreans. Euclid's Elements (c. 300 BC) provides several propositions and their proofs employing the golden ratio and contains the first known definition: A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser. Michael Maestlin, the first to write a decimal approximation of the ratio. The golden ratio was studied peripherally over the next millennium. Abu Kamil (c. 850–930) employed it in his geometric calculations of pentagons and decagons; his writings influenced that of Fibonacci (Leonardo of Pisa) (c. 1170–1250), who used the ratio in related geometry problems, though never connected it to the series of numbers named after him. Luca Pacioli named his book Divina proportione (1509) after the ratio and explored its properties including its appearance in some of the Platonic solids. Leonardo da Vinci, who illustrated the aforementioned book, called the ratio the sectio aurea ('golden section'). 16th-century mathematicians such as Rafael Bombelli solved geometric problems using the ratio.

German mathematician Simon Jacob (d. 1564) noted that consecutive Fibonacci numbers converge to the golden ratio; this was rediscovered by Johannes Kepler in 1608. The first known decimal approximation of the (inverse) golden ratio was stated as "about 0.6180340" in 1597 by Michael Maestlin of the University of Tübingen in a letter to Kepler, his former student. The same year, Kepler wrote to Maestlin of the Kepler triangle, which combines the golden ratio with the Pythagorean theorem. Kepler said of these: Geometry has two great treasures: one is the Theorem of Pythagoras, and the other the division of a line into extreme and mean ratio; the first we may compare to a measure of gold, the second we may name a precious jewel.18th-century mathematicians Abraham de Moivre, Daniel Bernoulli, and Leonhard Euler used a golden ratio-based formula which finds the value of a Fibonacci number based on its placement in the sequence; in 1843 this was rediscovered by Jacques Philippe Marie Binet, for whom it was named "Binet's formula". Martin Ohm first used the German term goldener Schnitt ('golden section') to describe the ratio in 1835. James Sully used the equivalent English term in 1875.

By 1910, mathematician Mark Barr began using the Greek letter Phi (φ) as a symbol for the golden ratio. It has also been represented by tau (τ), the first letter of the ancient Greek τομή ('cut' or 'section'). Between 1973 and 1974, Roger Penrose developed Penrose tiling, a pattern related to the golden ratio both in the ratio of areas of its two rhombic tiles and in their relative frequency within the pattern. This led to Dan Shechtman's early 1980s discovery of quasicrystals, some of which exhibit icosahedral symmetry.

A 2004 geometrical analysis of earlier research into the Great Mosque of Kairouan (670) reveals a consistent application of the golden ratio throughout the design. They found ratios close to the golden ratio in the overall layout and in the dimensions of the prayer space, the court, and the minaret. However, the areas with ratios close to the golden ratio were not part of the original plan and were likely added in a reconstruction. It has been speculated that the golden ratio was used by the designers of the Naqsh-e Jahan Square (1629) and the adjacent Lotfollah Mosque.

The Swiss architect Le Corbusier, famous for his contributions to the modern international style, centered his design philosophy on systems of harmony and proportion. Le Corbusier's faith in the mathematical order of the universe was closely bound to the golden ratio and the Fibonacci series, which he described as "rhythms apparent to the eye and clear in their relations with one another. And these rhythms are at the very root of human activities. They resound in man by an organic inevitability, the same fine inevitability which causes the tracing out of the Golden Section by children, old men, savages and the learned”. Le Corbusier explicitly used the golden ratio in his Modulor system for the scale of architectural proportion. He saw this system as a continuation of the long tradition of Vitruvius, Leonardo da Vinci's "Vitruvian Man", the work of Leon Battista Alberti, and others who used the proportions of the human body to improve the appearance and function of architecture. In addition to the golden ratio, Le Corbusier based the system on human measurements, Fibonacci numbers, and the double unit. He took suggestion of the golden ratio in human proportions to an extreme: he sectioned his model human body's height at the navel with the two sections in golden ratio, then subdivided those sections in golden ratio at the knees and throat; he used these golden ratio proportions in the Modulor system. Le Corbusier's 1927 Villa Stein in Garches exemplified the Modulor system's application. The villa's rectangular ground plan, elevation, and inner structure closely approximate golden rectangles.

Another Swiss architect, Mario Botta, bases many of his designs on geometric figures. Several private houses he designed in Switzerland are composed of squares and circles, cubes and cylinders. In a house he designed in Origlio, the golden ratio is the proportion between the central section and the side sections of the house.“Divine proportion”, a three-volume work by Luca Pacioli, was published in 1509. Pacioli, a Franciscan friar, was known mostly as a mathematician, but he was also trained and keenly interested in art. Divina proportione explored the mathematics of the golden ratio. Though it is often said that Pacioli advocated the golden ratio's application to yield pleasing, harmonious proportions, Livio points out that the interpretation has been traced to an error in 1799, and that Pacioli actually advocated the Vitruvian system of rational proportions. Pacioli also saw Catholic religious significance in the ratio, which led to his work's title.

Leonardo da Vinci's illustrations of polyhedra in Divina proportione have led some to speculate that he incorporated the golden ratio in his paintings. But the suggestion that his Mona Lisa, for example, employs golden ratio proportions, is not supported by Leonardo's own writings. Similarly, although the Vitruvian Man is often shown in connection with the golden ratio, the proportions of the figure do not actually match it, and the text only mentions whole number ratios.

Salvador Dalí, influenced by the works of Matila Ghyka, explicitly used the golden ratio in his masterpiece, The Sacrament of the Last Supper. The dimensions of the canvas are a golden rectangle. A huge dodecahedron, in perspective so that edges appear in golden ratio to one another, is suspended above and behind Jesus and dominates the composition.

A statistical study on 565 works of art of different great painters, performed in 1999, found that these artists had not used the golden ratio in the size of their canvases. The study concluded that the average ratio of the two sides of the paintings studied is 1.34, with averages for individual artists ranging from 1.04 (Goya) to 1.46 (Bellini). On the other hand, Pablo Tosto listed over 350 works by well-known artists, including more than 100 which have canvasses with golden rectangle and root-5 proportions, and others with proportions like root-2, 3, 4, and 6. According to Jan Tschichold, there was a time when deviations from the truly beautiful page proportions 2:3, 1:√3, and the Golden Section were rare. Many books produced between 1550 and 1770 show these proportions exactly, to within half a millimeter. According to some sources, the golden ratio is used in everyday design, for example in the proportions of playing cards, postcards, posters, light switch plates, and widescreen televisions.

Ernő Lendvai analyzes Béla Bartók's works as being based on two opposing systems, that of the golden ratio and the acoustic scale, though other music scholars reject that analysis. French composer Erik Satie used the golden ratio in several of his pieces. The golden ratio is also apparent in the organization of the sections in the music of Debussy's Reflets dans l'eau (Reflections in Water), from Images (1st series, 1905), in which "the sequence of keys is marked out by the intervals 34, 21, 13 and 8, and the main climax sits at the phi position".

The musicologist Roy Howat has observed that the formal boundaries of Debussy's La Mer correspond exactly to the golden section. Trezise finds the intrinsic evidence "remarkable" but cautions that no written or reported evidence suggests that Debussy consciously sought such proportions. Pearl Drums positions the air vents on its master's premium models based on the golden ratio. The company claims that this arrangement improves bass response and has applied for a patent on this innovation.

Johannes Kepler wrote that "the image of man and woman stems from the divine proportion. In my opinion, the propagation of plants and the progeniture acts of animals are in the same ratio". The psychologist Adolf Zeising noted that the golden ratio appeared in phyllotaxis and argued from these patterns in nature that the golden ratio was a universal law. Zeising wrote in 1854 of a universal orthogenetic law of "striving for beauty and completeness in the realms of both nature and art". In 2010, the journal Science reported that the golden ratio is present at the atomic scale in the magnetic resonance of spins in cobalt niobite crystals. However, some have argued that many apparent manifestations of the golden ratio in nature, especially in regard to animal dimensions, are fictitious. The description of this proportion as Golden or Divine is fitting perhaps because it is seen by many to open the door to a deeper understanding of beauty and spirituality in life. That's an incredible role for one number to play, but then again, this one number has played an incredible role in human history and the universe at large. When the ancients discovered 'Phi', they were certain they had stumbled across God's building block for the world.

Leonardo Da Vinci has long been associated with the golden ratio. This association was reinforced in popular culture in 2003 by Dan Brown's bestselling book "The Da Vinci Code." The plot has pivotal clues involving the golden ratio and Fibonacci series. In 2006, the public awareness of the association grew when the book was turned into a movie starring veteran actor Tom Hanks. Da Vinci's association with the golden ratio, known in his time as the Divine proportion, runs much longer and deeper. Da Vinci's illustrations appear in Pacioli's book "The Divine Proportion" Da Vinci created the illustrations for the book "De Divina Proportione" (The Divine Proportion) by Luca Pacioli. It was written in about 1497 and first published in 1509. Pacioli was a contemporary of Da Vinci's, and the book contains dozens of beautiful illustrations of three-dimensional geometric solids and templates for script letters in calligraphy.

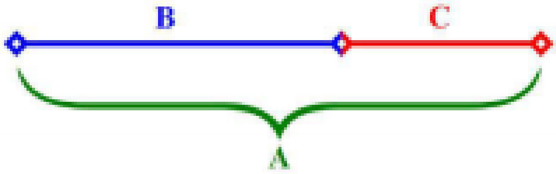
The frequency of appearance of the Golden Ratio in nature implies its importance as a cosmological constant and sign of being fundamental characteristic of the Universe. Except than Leonardo Da Vinci's Monalisa it appears on the sunflower seed head, flower petals, pinecones, pineapple, tree branches, shell, hurricane, tornado, ocean wave, and animal flight patterns. It is also very prominent on human body as it appears on human face, legs, arms, fingers, shoulder, height, eye-nose-lips, and all over DNA molecules and human brain as well. It is inevitable in ancient Egyptian pyramids and many of the proportions of the Parthenon. But very few of us are aware of the fact that it is part and parcel for constituting black hole's entropy equations, black hole's specific heat change equation, also it appears at Komar Mass equation of black holes and Schwarzschild–Kottler metric - for null-geodesics with maximal radial acceleration at the turning point of orbits. But here in this book the discussion is limited to the exhibition of mathematical aptitude of Golden Ratio a.k.a. the Divine Proportion, the Cosmological Constant and the Fundamental Constant of Nature.

Fibonacci Series (1 through 10)	Ratio derived from dividing this row's Fibonacci number by the previous # in the series	distance from phi 1.61803398874989...	Fibonacci Series (Continued 11-20)	Ratio derived from dividing this row's Fibonacci number by the previous # in the series	distance from phi 1.61803398874989...
0	n/a	n/a	55	1.61764705882353	0.00038692992636
1	n/a	n/a	89	1.61818181818182	-0.00014782943193
1	1.00000000000000	0.61803398874989	144	1.61797752808989	0.00005646066000
2	2.00000000000000	-0.38196601125011	233	1.61805555555556	-0.00002156680567
3	1.50000000000000	0.11803398874989	377	1.61802575107296	0.00000823767693
5	1.66666666666667	-0.04863267791678	610	1.61803713527851	-0.00000314652862
8	1.60000000000000	0.01803398874989	987	1.61803278688525	0.00000120186464
13	1.62500000000000	-0.00696601125011	1597	1.61803444782168	-0.00000045907179
21	1.61538461538462	0.00264937336527	2584	1.61803381340013	0.00000017534976
34	1.61904761904762	-0.00101363029773	4181	1.61803405572755	-0.00000006697766

A table exploring the relationship between the Golden Ratio and the Fibonacci series

II. ALGEBRA

By definition, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. Let's say, straight line A is divided into two segments, into B and C in such manner that:



A/B = B/C = Φ; or, B.B = A.C; But we know that A = B + C. So, B.B = [B + C].C; that is, B.B = B.C + C.C. Now if we divide this equation by C.C, [ie, C-Square], we will find that, B.B/C.C = B.C/C.C + C.C/C.C. So, (B/C)-Square = B/C + 1; That is, $\Phi^2 = \Phi + 1$ or, $[\Phi^2 - \Phi - 1] = 0$. Hence, $\Phi = 1.618033988749895\ldots$

Golden Ratio can be expressed in so many different ways. One of the most common expression is given below:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

From this expression it can be formulated into $\Phi = (1 + 1/\Phi)$, that is $\Phi^2 = (\Phi + 1)$ or, $[\Phi^2 - \Phi - 1] = 0$. Also another most common expression of Golden Ratio is:

$$\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}$$

From this expression it can be formulated into $\Phi = \sqrt{1 + \Phi}$; that is, $\Phi^2 = (\Phi + 1)$ or, $[\Phi^2 - \Phi - 1] = 0$.

As we can see the quadratic equation $[\Phi^2 - \Phi - 1] = 0$ gives the root value equal to the golden ratio, it can be written as $\Phi = [\Phi^2 - 1]$. Hence, $\Phi = (\Phi + 1).(\Phi - 1)$. ie, $1.618034 = 2.618034 \times 0.618034$. Another interesting fact about that equation is, $(\Phi + 1) = 2.618034 = \Phi^2$ & $(\Phi - 1) = 0.618034 = \frac{1}{\Phi}$. So, $(\Phi + 1).(\Phi - 1) = [\Phi^2.\frac{1}{\Phi}] = \Phi$.

Solution - A:

We know that, $[\Phi^2 - \Phi - 1] = 0$

Or, $2\Phi^2 - 2\Phi - 2 = 0$

Or, $2\Phi^2 - (+1)\Phi + (-1)\Phi - 2 = 0$

Or, $2\Phi^2 - (\sqrt{5} + 1)\Phi + (\sqrt{5} - 1)\Phi - 2 = 0$

Or, $\Phi.[2\Phi - (\sqrt{5} + 1)] + [(\sqrt{5} - 1)/2].[2\Phi - \frac{4}{(\sqrt{5} - 1)}] = 0$

Or, $\Phi.[2\Phi - (\sqrt{5} + 1)] + [(\sqrt{5} - 1)/2].[2\Phi - \frac{4(\sqrt{5} + 1)}{(\sqrt{5} + 1).(\sqrt{5} - 1)}] = 0$

Or, $\Phi.[2\Phi - (\sqrt{5} + 1)] + [(\sqrt{5} - 1)/2].[2\Phi - \frac{4(\sqrt{5} + 1)}{\{(\sqrt{5})\text{^}2 - (1)\text{^}2\}}] = 0$

Or, $\Phi.[2\Phi - (\sqrt{5} + 1)] + [(\sqrt{5} - 1)/2].[2\Phi - \frac{4(\sqrt{5} + 1)}{(5 - 1)}] = 0$

Or, $\Phi.[2\Phi - (\sqrt{5} + 1)] + [(\sqrt{5} - 1)/2].[2\Phi - 4(\sqrt{5} + 1)/(4)] = 0$

Or, $\Phi.[2\Phi - (\sqrt{5} + 1)] + [(\sqrt{5} - 1)/2].[2\Phi - (\sqrt{5} + 1)] = 0$

Or, $[2\Phi - (\sqrt{5} + 1)] \times [\Phi + \{(\sqrt{5} - 1)/2\}] = 0$

Or, $2.[\Phi - \{(\sqrt{5} + 1)/2\}] \times [\Phi + \{(\sqrt{5} - 1)/2\}] = 0$

Or, $[\Phi - \{(\sqrt{5} + 1)/2\}] \times [\Phi + \{(\sqrt{5} - 1)/2\}] = 0$

So, either $[\Phi - \{(\sqrt{5} + 1)/2\}] = 0$ or else, $[\Phi + \{(\sqrt{5} - 1)/2\}] = 0$

Which means, $\Phi = \frac{1 \pm \sqrt{5}}{2}$; that is, $+1.618034$ or, -0.618034.

Solution - B:

We Know That, $[\Phi^2 - \Phi - 1] = 0$

$\Rightarrow [\Phi^2 - 1.618\Phi + 0.618\Phi - 1] = 0$

$\Rightarrow \Phi(\Phi - 1.618) + 0.618(\Phi - \frac{1}{0.618}) = 0$

$\Rightarrow \Phi(\Phi - 1.618) + 0.618(\Phi - 1.618) = 0$

$\Rightarrow (\Phi - 1.618).(\Phi + 0.618) = 0$

ie, $(\Phi - 1.618) = 0$ or, $(\Phi + 0.618) = 0$

So, $\Phi = 1.618034$ or, $\Phi = -0.618034$.

Solution - C:

We Know That, $[\Phi^2 - \Phi - 1] = 0$

Or, $[ax^2 + bx + c] = 0$

Or, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

Where, $a = 1$, $b = -1$, $c = -1$

So, we can say that, $\Phi = \dfrac{1 \pm \sqrt{5}}{2}$;

that is, +1.618034 or, –0.618034.

The reason of getting two values are, by definition if we take the ratio of larger to shorter, then it will give us the positive root value, [ie, (Larger/Shorter) = 1.618034 = $\{(\sqrt{5} + 1)/2\}$]. But if we take the ratio of shorter to larger, then it will give us the negative root value, [ie, (Shorter/Larger) = 0.618034 = $\{(\sqrt{5} - 1)/2\}$]. Now, we can see that, (Larger/Shorter) × (Shorter/Larger) = 1. So, 1.618034 × 0.618034 = 1. Or in other way, $[(\sqrt{5} + 1)/2] \times [(\sqrt{5} - 1)/2] = [\{(\sqrt{5})^2 - 1^2\}/(2 \times 2)] = [(5 - 1)/4] = 1$.

Golden Ratio Φ in Fibonacci Sequence

It has been observed that Golden Ratio appears at Fibonacci Sequence as well. The Fibonacci sequence F_n is such that each number is the sum of the two preceding ones, starting from 0 & 1; that is, $F_n = F_{(n-1)} + F_{(n+2)}$. So, F_n= 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … F_n, $F_{(n+1)}$, $F_{(n+2)}$, … (up to infinity). One of the most frequently rediscovered facts about the Fibonacci Sequence is if we tabulate these numbers in a column, shifting the decimal point one place to the right for each successive number, the sum equals $1/F_{12}$, 1/89, as indicated in the next page.

Another fun fact of Fibonacci Sequence is (Last digit of F_{60}), (Last digit of F_{61}), (Last digit of F_{62}), … (up to infinity) = Fibonacci Sequence itself. So, the reason for bringing up this mysterious sequence is it has an uncanny relationship with the Golden ratio Φ.

Sum of:

0.0

0.01

0.001

0.0002

0.00003

0.000005

0.0000008

0.00000013

0.000000021

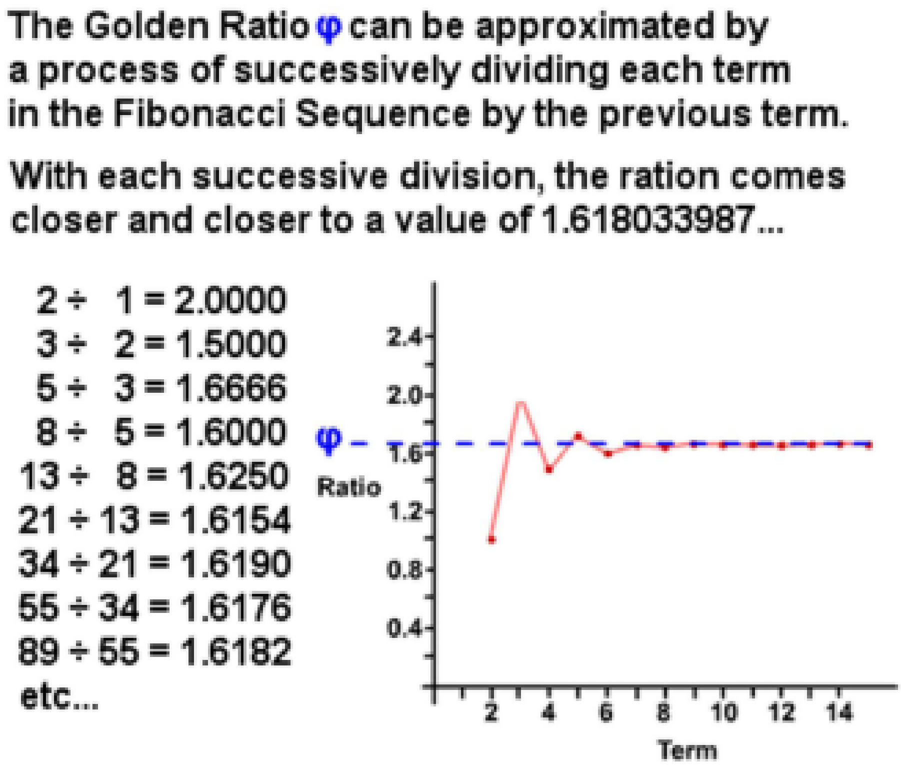
0.0000000034

0.00000000055

0.000000000089

… …ETC… …

0.01123595505618… = 1/89



It has been observed that, the golden ratio can be approximated by a process of successive dividing of each term in the Fibonacci Sequence by the previous term. And with each successive division, the result comes closer and closer to Φ. ie, $F_{(n+1)}/F_n = \Phi$. For example. 89/55 = 1.6181818181... very close to Φ; as shown in the graph above. Because, Fn = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,... A, B, C... (up to infinity). Say, B/A = X. So, C/B ≈ X, as well. Hence, B/A = C/B. But, C = (A + B). That is, B/A = (A + B)/B. Or, B/A = (A/B + 1). Which means that, X = (1/X + 1). Or, $X^2 = X + 1$; ie $[X^2 - X - 1] = 0$. Hence, X = Φ. But the most uncanny and eye soothing expression is given in the geometric series given below.

Say, $S = 1/2^{\Phi} + 1/2^{2\Phi} + 1/2^{3\Phi} + 1/2^{4\Phi} + 1/2^{5\Phi} + 1/2^{6\Phi} + 1/2^{7\Phi} + 1/2^{8\Phi} + 1/2^{9\Phi} + \ldots$

So, $2^{\Phi}.S = 1 + 1/2^{\Phi} + 1/2^{2\Phi} + 1/2^{3\Phi} + 1/2^{4\Phi} + 1/2^{5\Phi} + 1/2^{6\Phi} + 1/2^{7\Phi} + 1/2^{8\Phi} + 1/2^{9\Phi}\ldots$

Viz, $2^{\Phi}.S = 1 + S$. Which means that, $S = 1/(2^{\Phi} - 1)$. But very surprisingly;

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$1/2^{\Phi} + 1/2^{2\Phi} + 1/2^{3\Phi} + 1/2^{4\Phi} + 1/2^{5\Phi} + 1/2^{6\Phi} + 1/2^{7\Phi} + 1/2^{8\Phi} + 1/2^{9\Phi} + \ldots$ is equal to

$$= \cfrac{1}{2^0 + \cfrac{1}{2^1 + \cfrac{1}{2^1 + \cfrac{1}{2^2 + \cfrac{1}{2^3 + \cfrac{1}{2^5 + \cfrac{1}{2^8 + \cfrac{1}{2^{13} + \cfrac{1}{2^{21} + \cfrac{1}{2^{34} + \cfrac{1}{2^{55} + \cfrac{1}{2^{89} + \cdots}}}}}}}}}}}}$$

Now as long as we are talking about the geometric progression, let us form one.

$$\Phi^2 = \Phi^1 + 1$$

$$\Phi^3 = \Phi^2 + \Phi^1$$

$$\Phi^4 = \Phi^3 + \Phi^2$$

$$\Phi^5 = \Phi^4 + \Phi^3$$

$$\Phi^6 = \Phi^5 + \Phi^4$$

$$\Phi^7 = \Phi^6 + \Phi^5$$

$$\Phi^8 = \Phi^7 + \Phi^6$$

$$\Phi^9 = \Phi^8 + \Phi^7$$

Up to infinity.

We got these equations by multiplying Φ in both sides of the equation. Now if we add these equations up to the infinity then it will give us a geometric progression.

$\Phi^2 + \Phi^3 + \Phi^4 + \Phi^5 + \Phi^6 + \Phi^7 + \ldots = 1 + 2\,\Phi + 2\Phi^2 + 2\Phi^3 + 2\Phi^4 + 2\Phi^5 + 2\Phi^6 + 2\Phi^7 \ldots$

Viz, $\Phi^2 + \Phi^3 + \Phi^4 + \Phi^5 + \Phi^6 + \Phi^7 + \ldots = 1 + 2\,\Phi + 2[\Phi^2 + \Phi^3 + \Phi^4 + \Phi^5 + \Phi^6 + \ldots]$

Or, $1 + 2\Phi + [\Phi^2 + \Phi^3 + \Phi^4 + \Phi^5 + \Phi^6 + \Phi^7 + \Phi^8 + \Phi^9 + \ldots] = 0$

ie, $1 + \Phi + \Phi^2 + \Phi^3 + \Phi^4 + \Phi^5 + \Phi^6 + \Phi^7 + \Phi^8 + \Phi^9 + \ldots = -\Phi$

Means, $1 + \Phi.[1 + \Phi + \Phi^2 + \Phi^3 + \Phi^4 + \Phi^5 + \Phi^6 + \Phi^7 + \Phi^8 + \Phi^9 + \ldots] = -\Phi$

Or, $1 + \Phi.[-\Phi] = -\Phi$ Which gives us, $1 - \Phi^2 = -\Phi$. Or, $\Phi^2 = \Phi + 1$.

The reason for bringing up this geometric progression over here is, we can see;

$\Rightarrow \Phi^n = \Phi^{(n-1)} + \Phi^{(n-2)}$

$\Rightarrow \Phi^n = [\Phi^{(n-2)} + \Phi^{(n-3)}] + \Phi^{(n-2)}$

$\Rightarrow \Phi^n = 2\Phi^{(n-2)} + \Phi^{(n-3)}$

$\Rightarrow \Phi^n = 2[\Phi^{(n-3)} + \Phi^{(n-4)}] + \Phi^{(n-3)}$

$\Rightarrow \Phi^n = 3\Phi^{(n-3)} + 2\Phi^{(n-4)}$

$\Rightarrow \Phi^n = 3[\Phi^{(n-4)} + \Phi^{(n-5)}] + 2\Phi^{(n-4)}$

$\Rightarrow \Phi^n = 5\Phi^{(n-4)} + 3\Phi^{(n-5)}$

$\Rightarrow \Phi^n = 5[\Phi^{(n-5)} + \Phi^{(n-6)}] + 3\Phi^{(n-5)}$

$\Rightarrow \Phi^n = 8\Phi^{(n-5)} + 5\Phi^{(n-6)}$

$\Rightarrow \Phi^n = 8[\Phi^{(n-6)} + \Phi^{(n-7)}] + 5\Phi^{(n-6)}$

$\Rightarrow \Phi^n = 13\Phi^{(n-6)} + 8\Phi^{(n-7)}$

$\Rightarrow \Phi^n = 13[\Phi^{(n-7)} + \Phi^{(n-8)}] + 8\Phi^{(n-7)}$

$\Rightarrow \Phi^n = 21\Phi^{(n-7)} + 13\Phi^{(n-8)}$

$\Rightarrow \Phi^n = 21[\Phi^{(n-8)} + \Phi^{(n-9)}] + 13\Phi^{(n-8)}$

$\Rightarrow \Phi^n = 34\Phi^{(n-8)} + 21\Phi^{(n-9)}$

$\Rightarrow \Phi^n = 34[\Phi^{(n-9)} + \Phi^{(n-10)}] + 21\Phi^{(n-9)}$

$\Rightarrow \Phi^n = 55\Phi^{(n-9)} + 34\Phi^{(n-10)}$

$\Rightarrow \Phi^n = 55[\Phi^{(n-10)} + \Phi^{(n-11)}] + 34\Phi^{(n-10)}$

$\Rightarrow \Phi^n = 89\Phi^{(n-10)} + 55\Phi^{(n-11)}$

Which Means That, $\Phi^n = F_n\Phi^{[n-(n-1)]} + F_{(n-1)}.\Phi^{(n-n)} = F_n\Phi^1 + F_{(n-1)}.\Phi^0$

Viz, $\Phi^n = F_n\Phi + F_{(n-1)}$. Now, by dividing the both side of the equation by $F_{(n-1)}$ we get that, $\Phi^n/ F_{(n-1)} = F_n\Phi/ F_{(n-1)} + F_{(n-1)}/ F_{(n-1)}$ ie, $\Phi^n/ F_{(n-1)} = F_n/F_{(n-1)}\Phi + 1$. But we know that, $F_n/F_{(n-1)} = \Phi$. So, $\Phi^n/ F_{(n-1)} = \Phi.\Phi + 1$. That is, $\Phi^n/ F_{(n-1)} = \Phi^2 + 1$.

From this relation we can find any Fibonacci number $F_n = \Phi^{(n+1)}/(\Phi^2 + 1)$. Where the denominator $(\Phi^2 + 1)$ equals to nearly 3.61803398875… But it only works when n is greater than three (3).

Now, F_n = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,... A, B, C... (up to infinity). Say, B/A = X. So, C/B ≈ X, as well. Hence, B/A = C/B. But, C = (A + B). That is, B/A = (A + B)/B. Or, B/A = (A/B + 1). So, X = (1/X + 1). Or, $X^2 = X + 1$; That is, $[X^2 - X - 1] = 0$. Hence, X = Φ.

Again, by forming matrix, we can say that, $\begin{bmatrix} C \\ B \end{bmatrix} = \begin{bmatrix} (A+B) \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} A \\ B \end{bmatrix}$.

Here say, $Æ = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. So, the characteristic equation will be, $|Æ - \lambda.I| = 0$; where λ is eigenvalue of Æ, & I is a (2×2) identity matrix. So, $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - \lambda . \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0$; viz, $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$; or, $\begin{vmatrix} (1-\lambda) & 1 \\ 1 & -\lambda \end{vmatrix} = 0$; Means, $[-\lambda.(1 - \lambda) - (1\times 1)] = 0$. So, $[\lambda\text{^}2 - \lambda - 1] = 0$. And here Æ is a (2×2) binary matrix. And similar to this matrix, the highest probability of any non-trivial eigenvalues that show up in binary matrixes is (like this one), Φ. Furthermore the quadratic equation, $[\Phi^2 - \Phi - 1] = 0$; can be represented as, $\begin{vmatrix} 1 & \Phi \\ \Phi & (\Phi+1) \end{vmatrix} = 0$. And again, $\begin{vmatrix} \Phi & 1 \\ \Phi & (\Phi-1) \end{vmatrix} = 0$.

Another fun fact of the golden ratio is, if n is an even number then $\Phi^n + \frac{1}{\Phi^n}$ will be always an integer. And if n is an odd number then $\Phi^n - \frac{1}{\Phi^n}$ will be always an integer.

n is even:	n is odd:
n = 0, the sum equals 2	n = 1, the sum equals 1
n = 2, the sum equals 3	n = 3, the sum equals 4
n = 4, the sum equals 7	n = 5, the sum equals 11
n = 6, the sum equals 18	n = 7, the sum equals 29
n = 8, the sum equals 47	n = 9, the sum equals 76
n = 10, the sum equals 123	n = 11, the sum equals 199

III. GEOMETRY

From the inception of the idea of Golden Ratio, mathematicians all across the globe attempted to come up with equations corelating pi and phi. Personally, I figured two pi-phi relations: (i) $6\Phi^2 \approx 5\pi$ & (ii) $\Phi \approx \frac{7\pi}{5e}$; by myself. But nothing beats the pi-phi relation $\cos(\pi/5) = \Phi/2$. Here beneath goes the mathematical evidence of the claim.

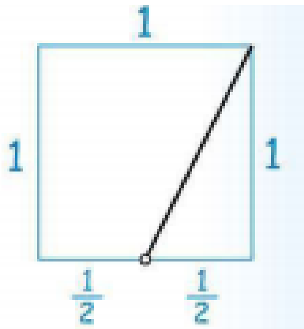
Let's say, $a = \cos(\pi/5)$ and $b = \cos(2\pi/5)$. Hence, $b = \cos(2\pi/5) = \cos(\pi/5 + \pi/5)$. Which means, term 'b' can be expressed as; $b = \cos(\pi/5).\cos(\pi/5) - \sin(\pi/5).\sin(\pi/5)$

$\Rightarrow b = \cos^2(\pi/5) - \sin^2(\pi/5)$

$\Rightarrow b = \cos^2(\pi/5) - [1 - \cos^2(\pi/5)]$

$\Rightarrow b = 2\cos^2(\pi/5) - 1 = 2[\cos(\pi/5)]^2 - 1$

$\Rightarrow \boxed{b = 2a^2 - 1}$... [equation (i)].



Again, $\cos(4\pi/5) = \cos[(5\pi - \pi)/5]$

$\Rightarrow \cos(4\pi/5) = \cos(\pi - \pi/5)$

$\Rightarrow \cos(4\pi/5) = \cos(\pi).\cos(\pi/5) - \sin(\pi).\sin(\pi/5)$

$\Rightarrow \cos(4\pi/5) = -\cos(\pi/5) = -a.$

As we know that, $\sin(\pi) = 0$ and $\cos(\pi) = -1$.

Hence, $-a = \cos(4\pi/5) = \cos(2\pi/5 + 2\pi/5)$.

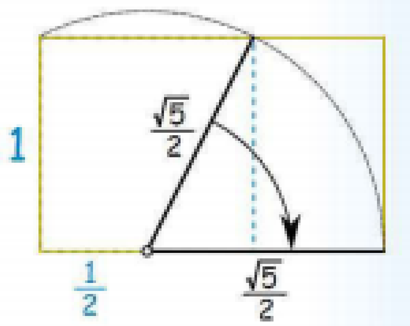
$\Rightarrow -a = \cos(2\pi/5).\cos(2\pi/5) - \sin(2\pi/5).\sin(2\pi/5)$

$\Rightarrow -a = \cos^2(2\pi/5) - \sin^2(2\pi/5)$

$\Rightarrow -a = \cos^2(2\pi/5) - [1 - \cos^2(2\pi/5)]$

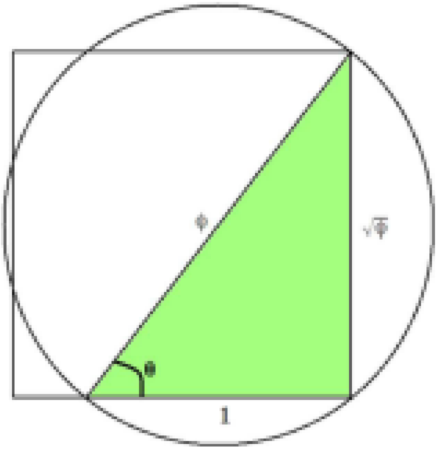
$\Rightarrow -a = 2\cos^2(2\pi/5) - 1 = 2[\cos(2\pi/5)]^2 - 1$

$\Rightarrow \boxed{-a = 2b^2 - 1}$... [equation (ii)].



Now if we deduct eqn. (ii) from eqn. (i), we get that; $(b + a) = (2a^2 - 1) - (2b^2 - 1)$. $\Rightarrow (b + a) = 2a^2 - 1 - 2b^2 + 1$. That is $(a + b) = 2a^2 - 2b^2 = 2(a^2 - b^2) = 2(a + b).(a - b)$. Which means, $(a - b) = (a + b)/[2(a + b)] = ½$ or, $\boxed{b = a - ½.}$ Putting this value in equation (i) gives us $a - ½ = 2a^2 - 1$, or, $2a^2 - a - 1 + ½ = 0$, that is $2a^2 - a - ½ = 0$. So, that is, $4a^2 - 2a - 1 = 0$. If we would put the value of b in equation (ii), we would've got, $-a = 2(a - ½)^2 - 1$. That is to say, $2(a^2 - a + ¼) - 1 + a = 0$.Which means, $4a^2 - 2a - 1 = 0$, the same. So, $a = \cos(\pi/5) = \frac{1 \pm \sqrt{5}}{4} = \Phi/2$.

Based on the concept of Pythagoras a right-angle triangle was made known as the Kepler Triangle, which is named after the German mathematician and astronomer



Johannes Kepler (1571–1630). The edge lengths in a precise geometric progression in which the common ratio is $\sqrt{\Phi}$; the geometric progression goes like $1 : \sqrt{\Phi} : \Phi$. Here, length of the hypotenuse of the right-angle triangle is Φ and so the other two arms have lengths of 1 and $\sqrt{\Phi}$. So, by Pythagoras $\Phi^2 = (\sqrt{\Phi})^2 + 1$; or, $\Phi^2 = \Phi + 1$ ie, $[\Phi^2 - \Phi - 1] = 0$.

The picture shown beside is a Kepler triangle. If Θ is the angle between hypotenuse Φ and base 1, then following relations can be drawn as well:
(i) $\sin\Theta = \sqrt{\Phi}/\Phi = 1/\sqrt{\Phi}$ (ii) $\cos\Theta = 1/\Phi$ (iii) $\tan\Theta = \sqrt{\Phi}$. Hence, we can say that, $\Theta = \sin^{-1}(1/\sqrt{\Phi}) = \cos^{-1}(1/\Phi) = \tan^{-1}\sqrt{\Phi} = 0.9$ rad $= 51.83^o$.
So, the other angle is $(180 - 90^o - 51.83^o) = 38.17^o = 2/3$ rad (roughly).

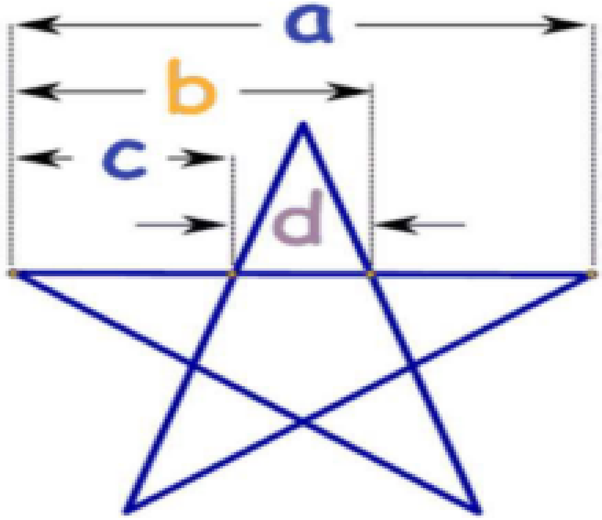
Another interesting fact of this diagram is, here we have a circle with a diameter of Φ and we have a square with sides of $\sqrt{\Phi}$. Though it is not possible to square a circle, we can see Sketch of the "Vitruvian Man" by Leonardo Vinci shows these two geometric figures have perimeter very close to each other. So, the circle and the square have closely equal perimeter. Now, perimeter of the square is four times its arms, viz. $4\sqrt{\Phi}$. And perimeter of the circle is 2π.radius $= \pi$.Diameter $= \pi.\Phi$. Hence, we can say, $\pi.\Phi \approx 4\sqrt{\Phi}$, ie, $\pi = 4/\sqrt{\Phi}$. It fit for an error that's less than 0.1%. Which brings us to another pi-phi relationship.

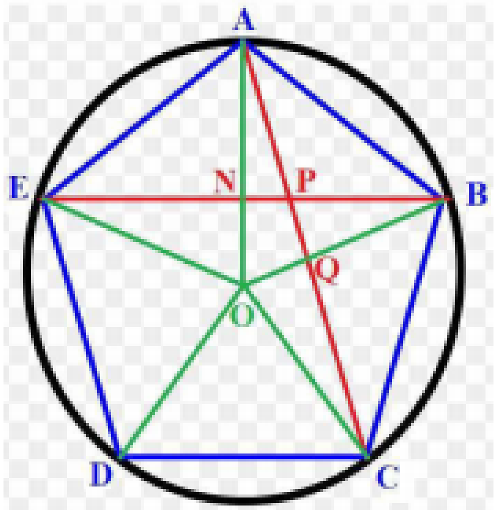
Now here in this segment of discussion we will get to know how we can draw the golden ratio as well as the geometric interpretation of it. Here is one way to draw a rectangle with the Golden Ratio. ***The figure is available in the previous page***. First draw a square of unit length, that is the length is one. Place a dot at half way along one side & draw a line from that point to an opposite corner. So, the line will have a length of $\sqrt{(1)^2 + (½)^2} = \sqrt{[1 + ¼]} = \sqrt{5/4} = \sqrt{5}/2$; Now either we add this value with ½ or we deduct this value from ½ to get the golden ratio. So, we turn that line so that it runs along the square's side and then we extend the square to be a rectangle with the Golden Ratio as shown in diagram. Notice that the arm of the rectangle is $(½ + \sqrt{5}/2)$ while the additional extended portion is $(½ - \sqrt{5}/2)$. So, we get both Φ and $-1/\Phi$ from this diagram.

PROBLEM # 1

Another interesting geometrical expression of the golden ratio can be obtain at perfect pentagon shown below. Here in this diagram a/b = b/c = c/d = Φ.

Then to prove the claim we need to change the diagram a little bit. We need to draw a polygon inscribed inside a circle consisting five arms. Besides, the assumptions will be all the five arms of the 'polygon' will have equal length. Suppose ⌂ABCDE is the polygon. So here, AB = BC = CD = DE = AE. Hence, ∠A = ∠B = ∠C = ∠D = ∠E.





And all the angles of the pentagon are equal to be: ϴ = [{(n – 2)×180°}/n]. As here n = 5, so; ϴ = [{(5 – 2)×180°}/5] = [(3×180°)/5] = (3×36°) = 108°. Having said that, it is noticeable that a perfect pentagon will inscribe inside a circle and will divide the circle into 360°/5 = 72°. Again, (72° + 108°) = 180°, also 72° = (36°×2), as well as, 108° = (36°×3) & 180° = (36°×5). Also we know that cos(π/5) = cos(36°) = Φ/2.

So over here, AB = BC = CD = DE = AE And, ∠A = ∠B = ∠C = ∠D = ∠E = 108°. As well as, AO = BO = CO = DO = EO, where O is the center of the circle. Join B & E. Line BE intersects line AO at point N. So, AN⊥BE, as well as NE = NB = ½BE. Join A & C. Line AC intersects line BE at point P and line BO at point Q. Also, BQ⊥AC; which means, AQ = CQ = ½AC. So, all we need to prove that, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \Phi$.

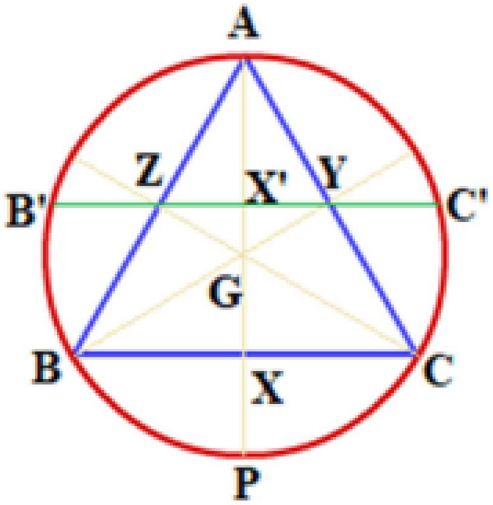
Now at ΔAEB; AE = BE. And as, ∠BAE = 108°
So other two angles; ∠AEB = ∠ABE = (180° – 108°)/2 = 72°/2 = 36°.
In ΔAEN; ∠ANE = 90°, ∠NAE = 108°/2 = 54°.
So ∠AEN will be equal to (180° – 90° – 54°) = 36°.
Now, ∠AEN = ∠AEB. So, in ΔAEN; cos∠AEN = NE/AE
viz, 2cos∠AEN = 2NE/AE. So, 2cos(36°) = (NE+NE)/AE
viz, 2cos(π/5) = (NE+BN)/AE. That is, 2×Φ/2 = BE/AE; viz, BE/AE = Φ.

So, now all we will need to prove is AE = PE to prove the pentagon relation stated before. If we can prove AE = PE, then BE/PE will be equal Φ.

Now at ΔABC; AB = BC. As, ∠ABC = 108°
So other two angles; ∠BAC = ∠BCA = (180° – 108°)/2 = 72°/2 = 36°.
In ΔAPB; ∠BAP = ∠BAC = 36° & ∠ABP = ∠ABE = 36°.
Which means, AP = BP & ∠APB = (180° – 36° – 36°) = (180° – 72°).
So, ∠APE = (180° – ∠APB) = [180° – (180° – 72°)] = 72°.
Now in ΔAEP; ∠APE = 72° & ∠AEP = ∠AEB = 36°;
Hence, ∠PAE = [180° – ∠APE – ∠AEP] = [180° – 72° – 36°] = 72°.
So in ΔAEP; ∠PAE = ∠APE = 72°.
So we can say, AE = PE. That is [BE]/[AE] = [BE]/[PE] = Φ. That is, $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \Phi$.
So, we can conclude by saying that the line BE is divided at golden ratio at point P.

PROBLEM # 2

Golden ratio can be expressed geometrically via an equilateral triangle inscribed inside a circle as well. In the figure below ΔABC is an equilateral triangle inscribed in a circle with center G and radius of R = AG = BG = CG. Now extend AG, that intersects BC at point X, & extend BG, that intersects AC at point Y, and extend CG, that intersects AB at point Z. Join Z & Y and extend in both directions to intersect the circle at point B' & C'. From this construction we will see that the ratio, ZY/B'Z = ZY/C'Y = B'Y/ZY = C'Z/ZY = Φ.



Let us assume, ZY = a, & YC' = b. We need to prove that a/b = (a + b)/a.

Now here, at ΔABC, AB = BC = CA. Again, AG = BG = CG = 2GX = 2GY = 2GZ. Also, AX = BY = CZ. So, AX⊥BC, BY⊥AC and CZ⊥AB. We know that, points X, Y & Z are midpoints of BC, CA, and AB respectively. Hence, AZ = BZ = BX = CX = CY = AY = ½AB = ½BC = ½CA. Now BC||B'C'. So, ΔAZY; ∠AZY = ∠ABY & ∠AYZ = ∠ACB & ∠ZAY = ∠BAC. Which means, ΔAZY is also equilateral as all these angles are 60°. Means, AZ = ZY = AY = a.

Now, as product of segments of two intersecting cords of a circle are equal. So, at point Y; [B'Y.YC' = AY.YC]. That is, (B'Z + ZY).YC' = AY.YC. As B'Z = YC'

So, (YC' + ZY).YC' = AY.YC = AY.AY = ZY.ZY; [as ΔAZY is equilateral].
Assume that, YC' = b & ZY = a;
Hence, (b + a).b = (a.a).
$\Rightarrow$ a/b = (b + a)/a
$\Rightarrow$ a/b = b/a + a/a
$\Rightarrow$ a/b = b/a + 1.

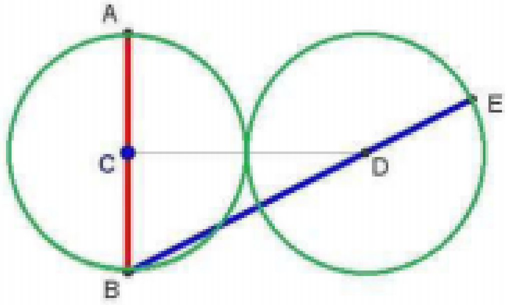
The expression beside may formulated into $\Phi = (1 + \frac{1}{\Phi})$, that is, $\Phi^2 = (\Phi + 1)$ or, $[\Phi^2 - \Phi - 1] = 0$ (proved).

Now in ΔABP; ∠BAP = 30º, ∠ABP = 90º, so, ∠APB = 60º. Now, sin∠APB = AB/AP
Say the arms of the equilateral triangles are of length 'a' and radius of circle is R.
Hence, sin∠APB = sin60º = a/2R; viz, R = a/(2sin60º).
Now as sin60º = $\sqrt{3}/2$, so, R = $a/\sqrt{3}$.
Also in ΔABX, sin∠ABX = sin60º = $\sqrt{3}/2$ = AX/AB
So, $\sqrt{3}/2$ = (AG + GX)/AB = (R + ½R)/a = (2R + R)/2a = 3R/2a; viz, R = $a/\sqrt{3}$.
Now, PX' = (AP – AX') = (AP – AZ.sin∠AZX)
So, PX' = (2R – ½AB.sin60º) = $(2a/\sqrt{3} - ½a.\sqrt{3}/2) = (2a/\sqrt{3} - a.\sqrt{3}/4)$;
ie, PX' = $a(2/\sqrt{3} - \sqrt{3}/4) = a[(8-3)/4\sqrt{3}] = 5a/4\sqrt{3}$.
That is, GX' = PX'– PG = PX'– R = $5a/4\sqrt{3} - a/\sqrt{3} = a/4\sqrt{3}$

Now in ΔB'X'G; $(B'X')^2 = (B'G)^2 - (GX')^2$
So, $(B'X')^2 = R^2 - (a/4\sqrt{3})^2 = (a/\sqrt{3})^2 - (a/4\sqrt{3})^2 = [a^2/3 - a^2/48] = 5a^2/16$.
Hence, B'X' = $(\sqrt{5}a/4)$. But, X'Y = YC' = ½a.
So, B'Y/YC' = (B'X' + X'Y)/YC' = $[\frac{\sqrt{5}a}{4} + \frac{a}{2}]/(\frac{a}{2})$.
So the ratio becomes equal to $(1 + \sqrt{5})/2 = 1.618034 = \Phi$ (proved).

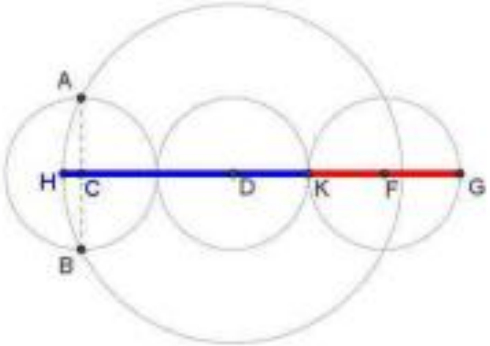
PROBLEM # 3

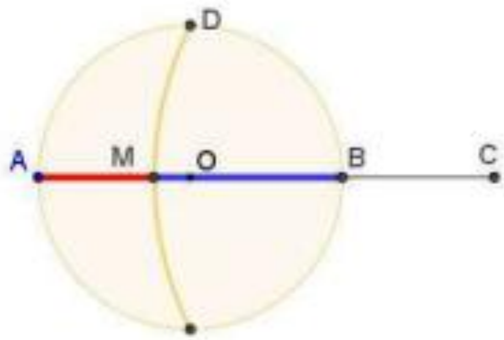
The golden ratio is available in numerous other images, another example can be the diagram shown below. Here, two equal circles with center C & D are tangent to each other. AB is the diameter of the circle with center C. CD⊥AB. Join B & D. BD crosses the circle with center D in two points, let E be further one from B.



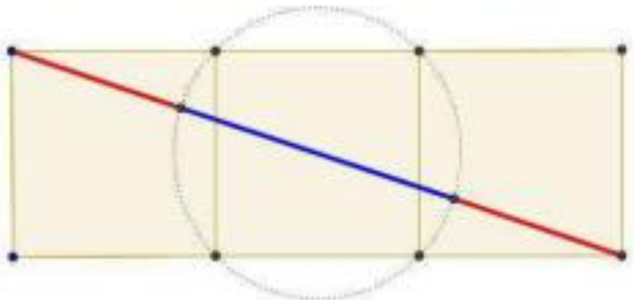
Let the radius of both circles r.
In ΔBCD, $(BD)^2 = (CD)^2 + (BC)^2$.
But, AC = BC = ½AB = DE = ½CD = r.
$\Rightarrow (BD)^2 = (2r)^2 + (r)^2$. $\Rightarrow (BD)^2 = 4(r)^2 + (r)^2$
$\Rightarrow (BD)^2 = 5(r)^2$. Suggests, BD = $\sqrt{5}$r.
Now, BE = BD + DE. So, BE = $(\sqrt{5}r + r)$.
Again AB = 2r. Therefore, the ratio BE/AB = $(\sqrt{5}r + r)/2r = (1 + \sqrt{5})/2 = 1.618034 = \Phi$.

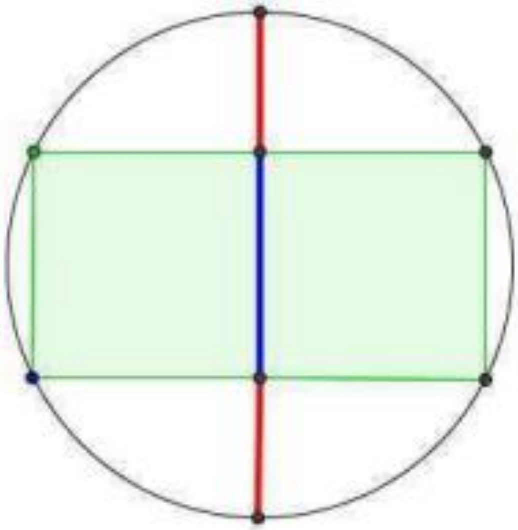
Not only this but also this concept can be nicely modified into a construction with four circle which is shown in the diagram below (left). As well as another most straightforward construction of the golden ratio with this concept has been devised by Nguyen Thanh Dung shown in the diagram below (right).

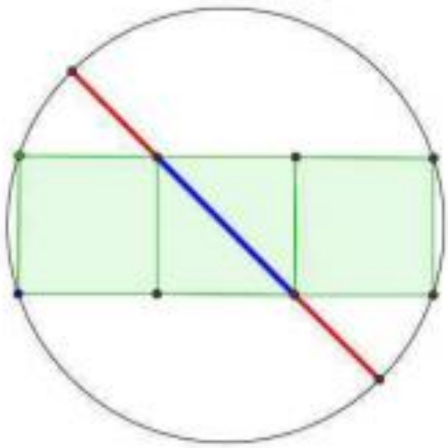




Tran Quang Hung has devised another configuration of a 1×3 rectangle with a circle that produces golden ratio. But There is a not immediately obvious relation between the case of 1×2 and 1×3 rectangles. If we consider the arms of the square of the diagram to be 'a', then the red plus blue line equals $\sqrt{(3a)^2 + (a)^2} = a\sqrt{10}$; while the blue one becomes equals to the diameter $\sqrt{(a)^2 + (a)^2} = a\sqrt{2}$ (1st-bottom-left). So, the ratio becomes equal to $\sqrt{5}$.





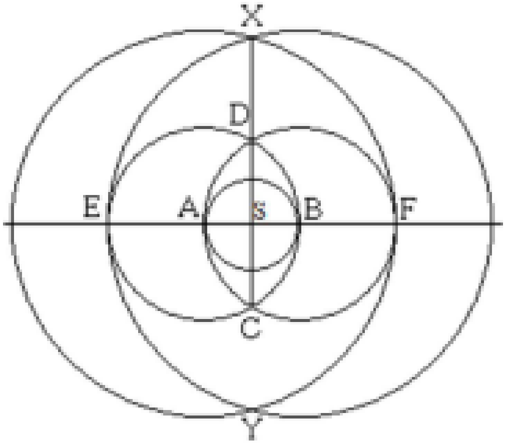


In all these three diagrams the above-mentioned relationship can be observed. And as we know that the value of $\Phi = \frac{1 \pm \sqrt{5}}{2}$; therefore, golden ratio is found in all these three diagrams as well.

Golden ratio has also observed in the constructions that involves a rhombus and a regular hexagon. Before going into that discussion it will be better to converse about another very elegant way of obtaining the golden ratio, offered in a (2002) article by K. Hofstetter. It's shown in the diagram provided below. Here, it will be convenient to denote S(R) the circle with center S through point R. For the construction, let A and B be two points. Circles A(B) and B(A) intersect in C and D and cross the line AB in points E and F. Circles B(E) and A(F) intersect in X and Y, as in the diagram. Because of the symmetry, points X, D, C, Y are collinear. The fact is CX/CD = Φ.

Assume for simplicity that AB = 2. Then CD = $2\sqrt{3}$, & CX = $\sqrt{15} + \sqrt{3}$. Hence, the ratio of CX & CD:

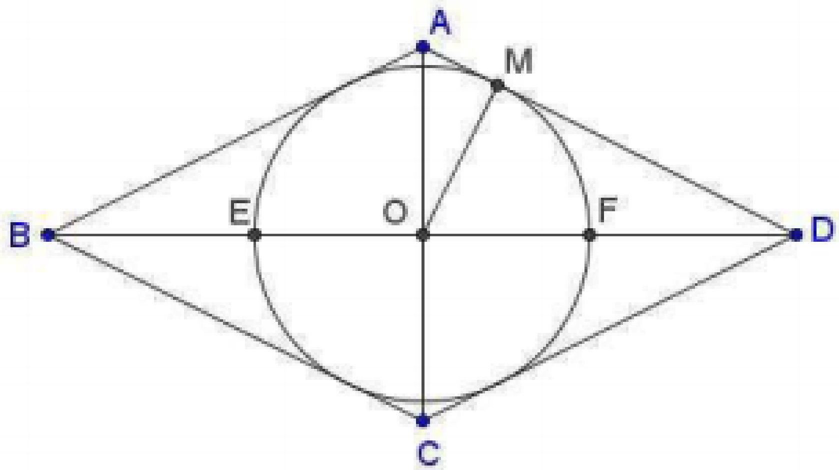
$$(CX)/(CD) = (\sqrt{15} + \sqrt{3})/2\sqrt{3}$$
$$= (\sqrt{5} + 1)/2$$
$$= \Phi.$$



Notice that the whole construction can be accomplished with compass only. This much simplicity as well as diversity has made golden ratio this much widespread and this is the reason of calling it in different other names like the golden mean or golden section. Similarly some other names include extreme mean ratio, medial section, divine cut proportion, divine section, golden cut, golden proportion and golden number. So, now we will discuss how this ratio has also observed in constructions involving a rhombus and a regular hexagon.

PROBLEM # 4

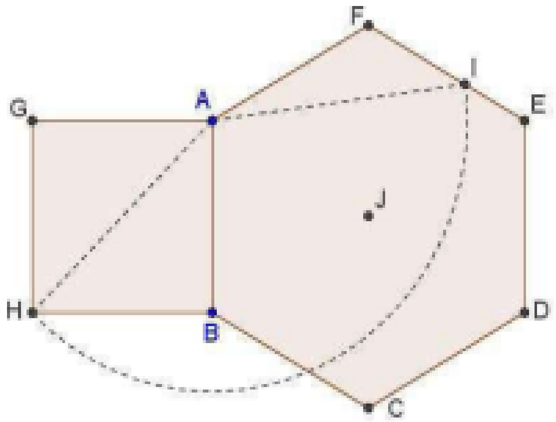
Let, ABCD is a rhombus with 2AC = BD. The inscribed circle has a center O. Also, E and F are the points of intersection of the circle with BD. Then, the point F divides DE in the golden ratio.



Now, let M be the point of tangency of (O) with AD. So, OM⊥AD. Hence, ΔMOD & ΔAOD are similar as ∠ADO = ∠MDO Therefore, (MD)/(MO) = (OD)/(OA) = 2; viz, MD = 2(OM) = EF. From the property of a tangent, DM^2 = DF·DE.
Or, EF^2 = DF·DE. Therefore, (FD)/(FE) = (EF)/(ED), as required.

PROBLEM # 5

Tran Quang Hung has devised another configuration of golden ratio Φ in a hexagon. Square ABHG is constructed outside the hexagon ABCDEF. A circle with center at A, radius AH cuts EF at I in golden ratio as shown in the diagram below.



AB = BC = CD = DE = EF = AF = GA = BH = a Therefore, AH = AI = $\sqrt{2}$a.
Set α = ∠FAI, β = ∠AIF.
Now by applying the Law of Sines in ΔAIF:
AF/sinβ = FI/sinα = AI/sin120°.
Now as AF = a and AI = $\sqrt{2}$a
So, this can be rewritten as:
$\frac{a}{\sin\beta}$ = ($\sqrt{2}$a)/($\sqrt{3}$/2) = (2$\sqrt{2}$)a/$\sqrt{3}$ = (2$\sqrt{6}$)a/3
Thus we can say:
sinβ = 3/(2$\sqrt{6}$) = 6/(4$\sqrt{6}$) = $\sqrt{6}$/4.

Hence, cosβ = $\sqrt{1-\ (sin\beta)^2}$ = $\sqrt{1-6/16}$ = $\sqrt{10}$/4. Now, (α+β) = 60°. Therefore, sinα = sin(60°−β) = sin60°·cosβ − cos60°·sinβ.

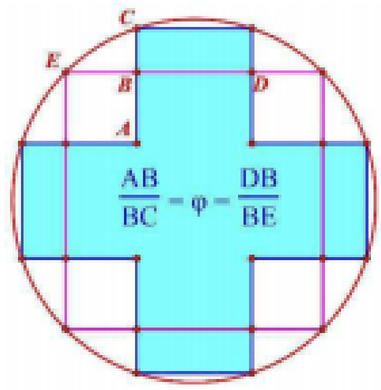
Thus, sinα = $[\frac{\sqrt{3}}{2}.\frac{\sqrt{10}}{4} - \frac{1}{2}.\frac{\sqrt{6}}{4}] = [\frac{\sqrt{30}}{8} - \frac{\sqrt{6}}{8}] = \frac{\sqrt{6}}{8}.(\sqrt{5}-1)$

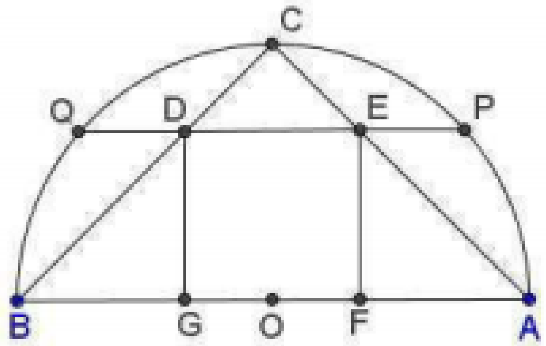
Now, AF/sinβ = FI/sinα; or, (2$\sqrt{6}$)a/3 = FI/$[\frac{\sqrt{6}}{8}.(\sqrt{5}-1)]$

Or, FI/a = ($\sqrt{5}$ − 1)/2 viz, a/FI = FE/FI = ($\sqrt{5}$ + 1)/2 = Φ.

Not only these there are several other numerous geometrical figures where golden ratio is observed. The following is a new invention of Bui Quang Tuan. In the diagram given in the next page the cross consists of five equal square. Here, let S be the side of the inscribed square, C the side of any of the five squares that compose the cross, then $S^2 = 5C^2$. From this expression the following relationship can be obtained as mentioned in the image.

In 2015 Tran Quang Hung has found once more the golden ratio in a combination of a semicircle, a square, & a right isosceles triangle. Given a right isosceles triangle ABC and its circumcircle, inscribed a square DEFG with a side FG along the hypotenuse AB. Let the side DE extended beyond E intersect the circumcircle at P. Then the point E divides DP in the golden ratio.

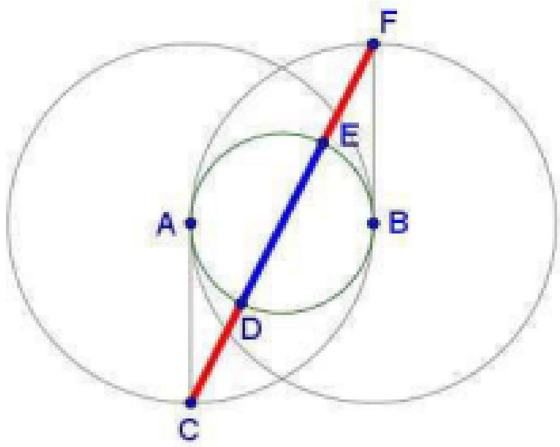




From the similarity of the isosceles right triangles ΔDEC and ΔAEF, we have (DE)/(CE) = (AE)/(EF).It thus follows; $DE^2 = DE \cdot EF = AE \cdot EC$. If the line DE intersects the semicircle again at Q, then EQ = DP. By the intersecting chords theorem, $AE \cdot CE = EP \cdot EQ = EP \cdot DP$. Therefore, $DE^2 = EP \cdot DP$, meaning that E divides DP in the golden ratio.

PROBLEM # 6

The following (right) construction of the golden ratio Φ has appeared in the Mathematical Gazette, volume 101, number 551, July 2017, page 303 constructed by John Molkach. There are two unit circles (A) & (B). The circle (O) has a 2R diameter of AB and tangent to both circles. Vertical segments AC & BF are tangent to circle B & circle A, respectively. So, AB = AC = BF = 1. CF crosses (O) in D and E, as shown in the diagram.



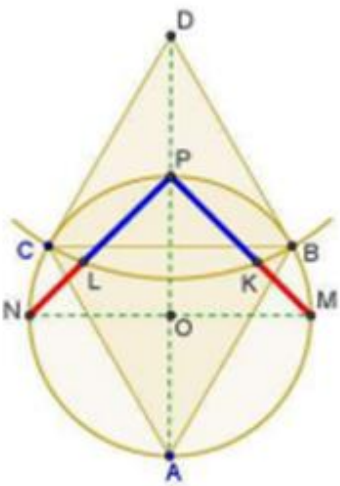
John proves that CE = Φ. To prove that let us consider, CE = x. Then, by Intersecting Secants rules $CA^2 = CD \times CE$. Or, $(1)^2 = (x - 1).x$ Viz $x^2 - x - 1 = 0$.

PROBLEM # 7

We are going to conclude our discussion for this chapter with another example of Tran Quang Hung. Let ABC be an equilateral triangle inscribed in circle (O). D is reflection of A through BC. MN is diameter of (O) parallel to BC. &AD meets (O) again at P. Then, circle (D) and passing through B, C divides PM, PN in golden ratio.

In this diagram, OB = OC = OA = ON = OM = OP = PD = PC = PB = R (say).

So, BC = CD = BD = AB = AC = DL = DK = $\sqrt{3}$R (as discussed before). Again,



NP = MP = $\sqrt{R^2 + R^2} = \sqrt{2}$R. In ΔDLP, <DPL = 135°
Say, PL = x. For a triangle with sides a, b, c & angle μ opposite to c, as per law of cosine:

$c^2 = a^2 + b^2 - 2ab\cdot\cos(\mu)$
viz, $DL^2 = PD^2 + PL^2 - 2.PD.PL.\cos 135°$
so, $(\sqrt{3}R)^2 = (R)^2 + (x)^2 - 2.R.x.\cos(180° - 45°)$
Now, as $\cos 45° = \frac{1}{\sqrt{2}}$
so, $(x)^2 + \sqrt{2}.R(x) - 2(R)^2 = 0$
Or, $2x = -\sqrt{2}.R \pm\sqrt{2R^2 + 8R^2}$
With positive value, $x = \frac{\sqrt{10}-\sqrt{2}}{2}.R = \frac{\sqrt{5}-1}{\sqrt{2}}.R$ = PL.

Now, NL = NP – PL = $\sqrt{2}R - \frac{\sqrt{5}-1}{\sqrt{2}}.R = \frac{3-\sqrt{5}}{\sqrt{2}}.R$.

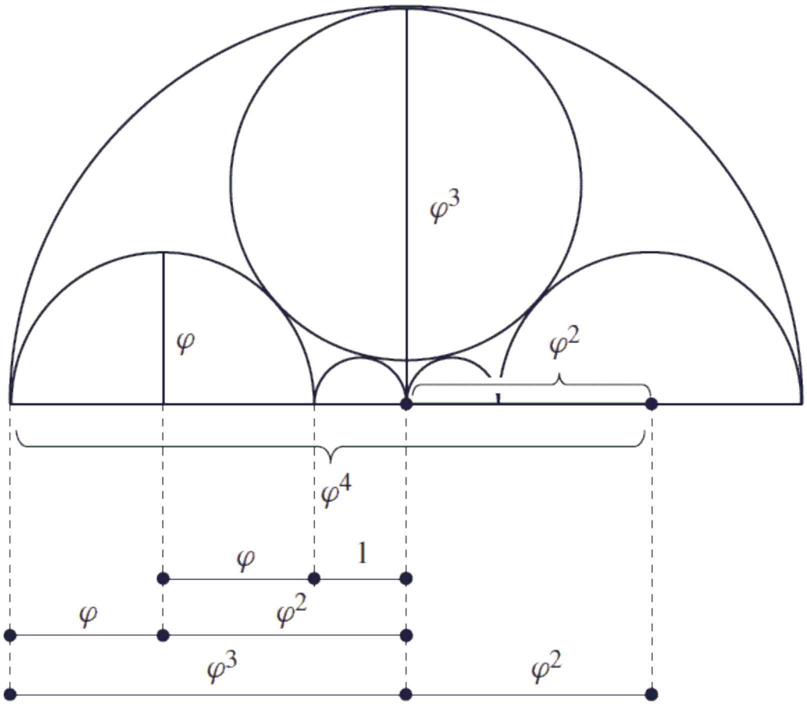
Now by taking ratio, PL/NL = $[\frac{\sqrt{5}-1}{\sqrt{2}}.R]/[\frac{3-\sqrt{5}}{\sqrt{2}}.R] = \frac{\sqrt{5}-1}{3-\sqrt{5}}$

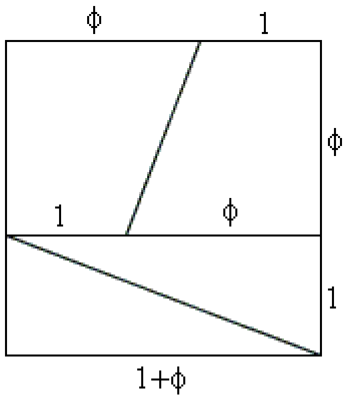
viz, PL/NL = $\frac{\sqrt{5}-1}{3-\sqrt{5}}.\frac{\sqrt{5}+1}{\sqrt{5}+1}.\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{(5-1).(\sqrt{5}+1)}{(3-\sqrt{5}).(5+2\sqrt{5}+1)} = \frac{(4).(\sqrt{5}+1)}{2(3-\sqrt{5}).(3+\sqrt{5})} = \frac{(4).(\sqrt{5}+1)}{2(3^2-\sqrt{5}^2)}$

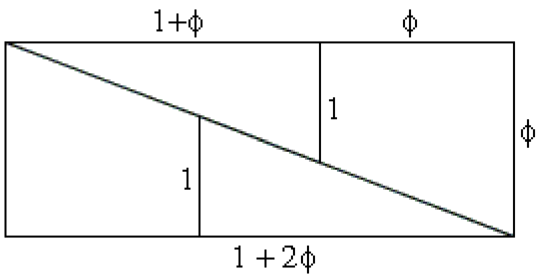
Thus we can prove that, PL/NL = $\frac{(\sqrt{5}+1)}{2} = \Phi$.

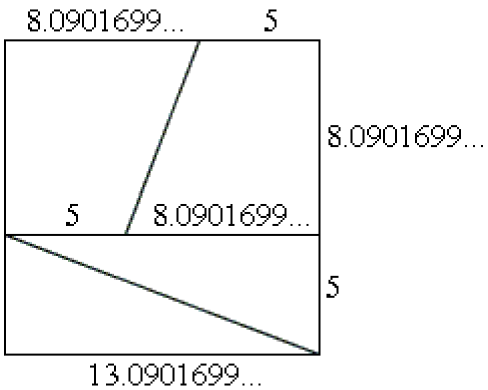
It is not so much that the golden ratio is "related to a fractal," as fractal patterns are based on any number. Fractal patterns created using golden ratio, however, are optimized in a way that does not occur with any other number. As an example, in the image below the fractal pattern expands using the golden ratio. These things may not seem very significant, but one could reach an understanding of these rates and ratios past the numbers, yet they appear almost perfectly in these huge megalithic archaeological structures.

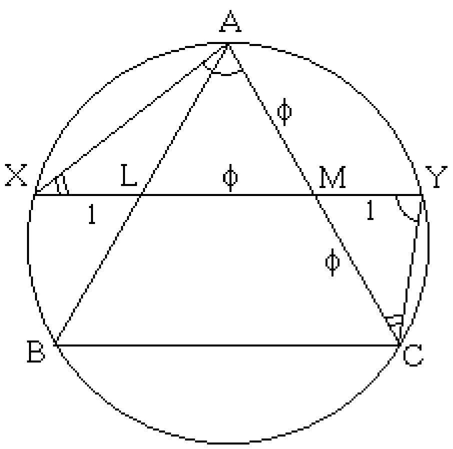
TASK FOR THE READERS:

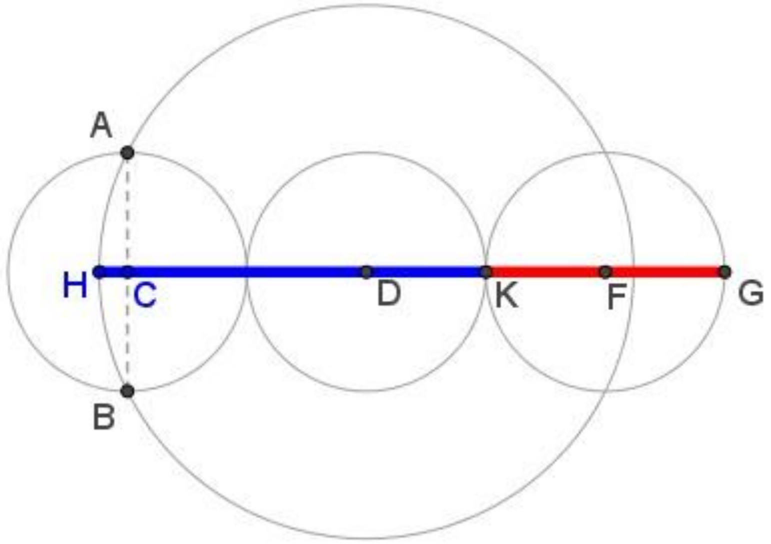


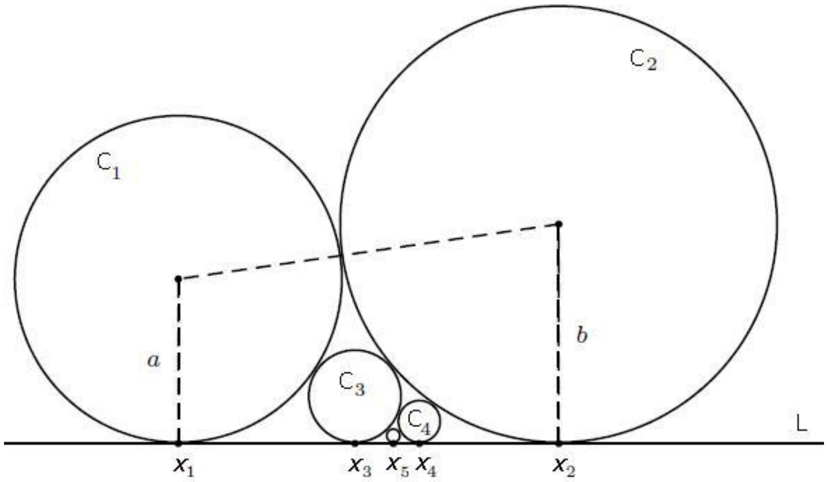


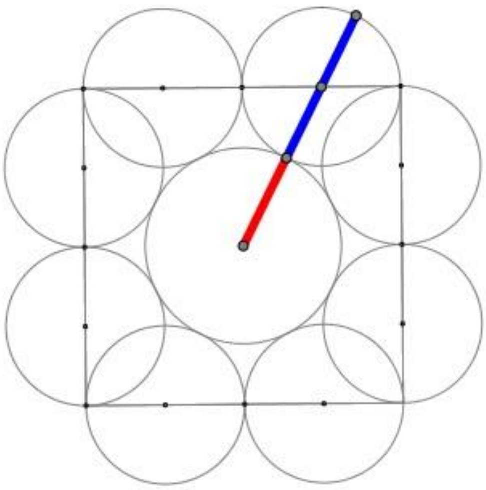


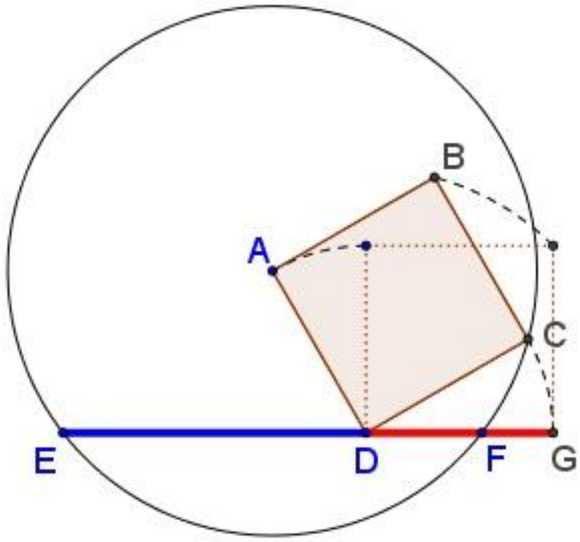


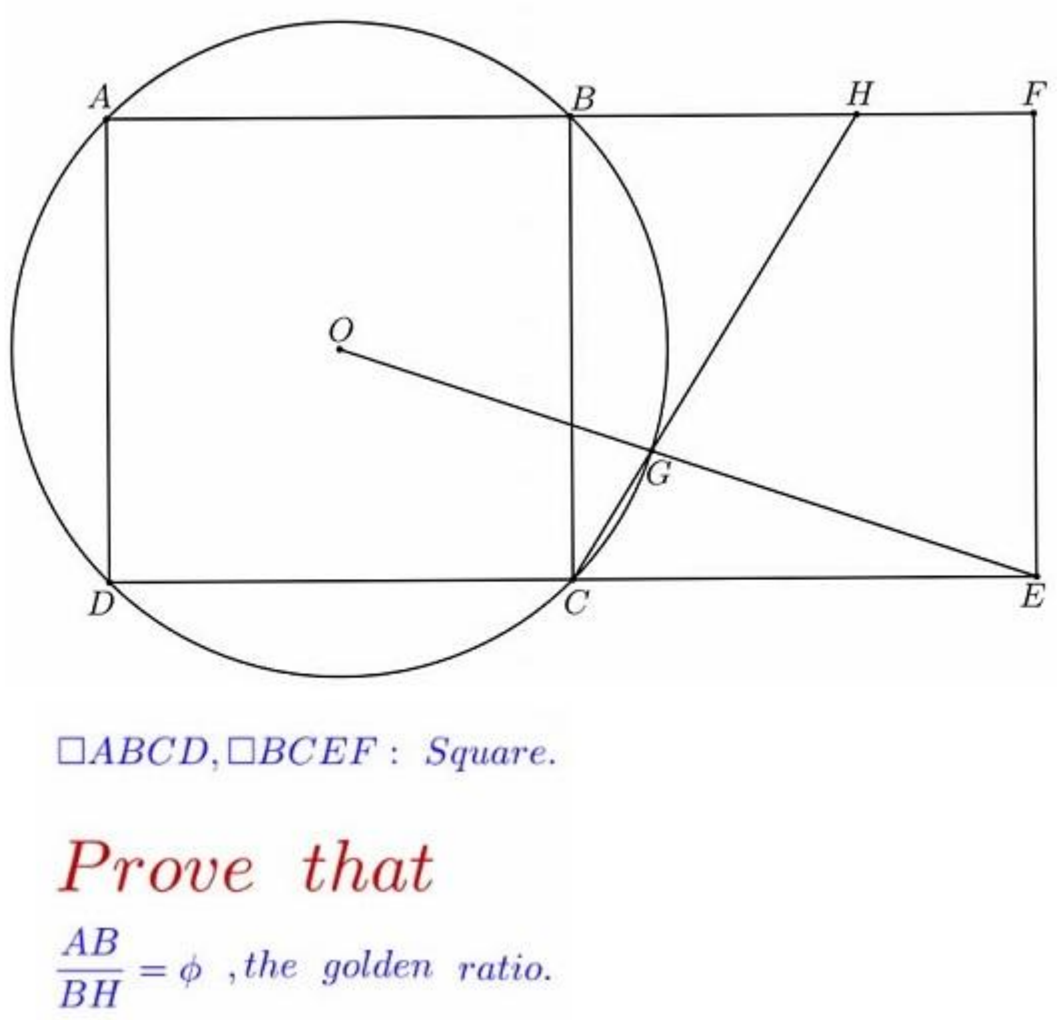


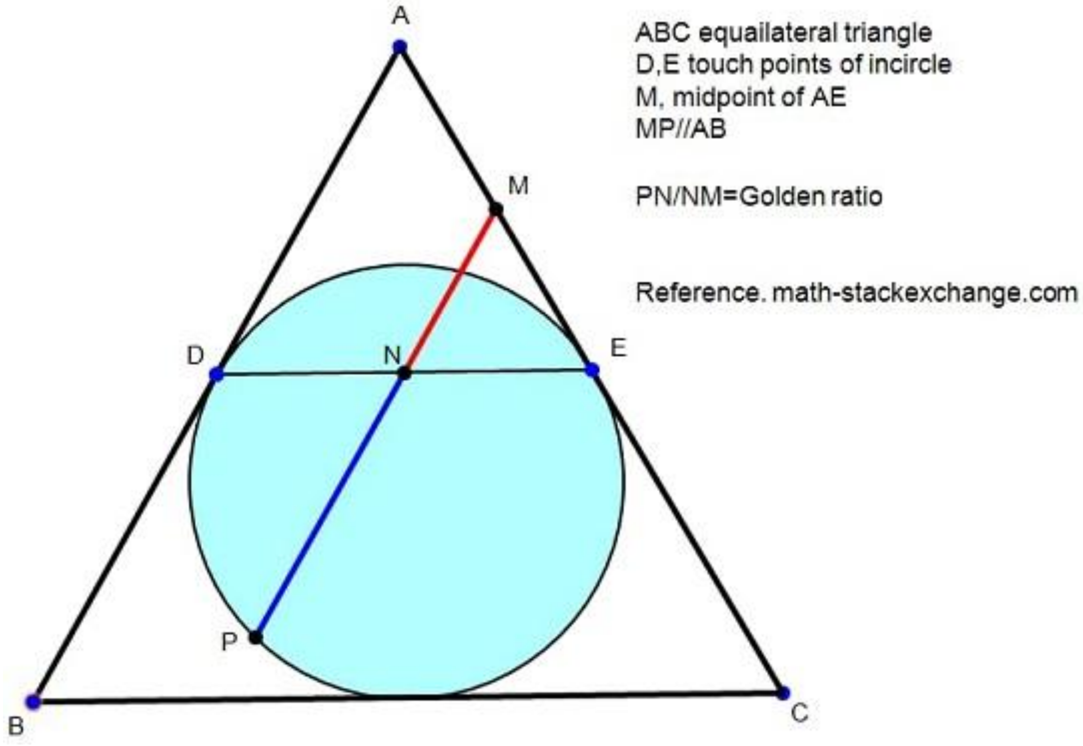

A
H
C
D
K
F
G
B

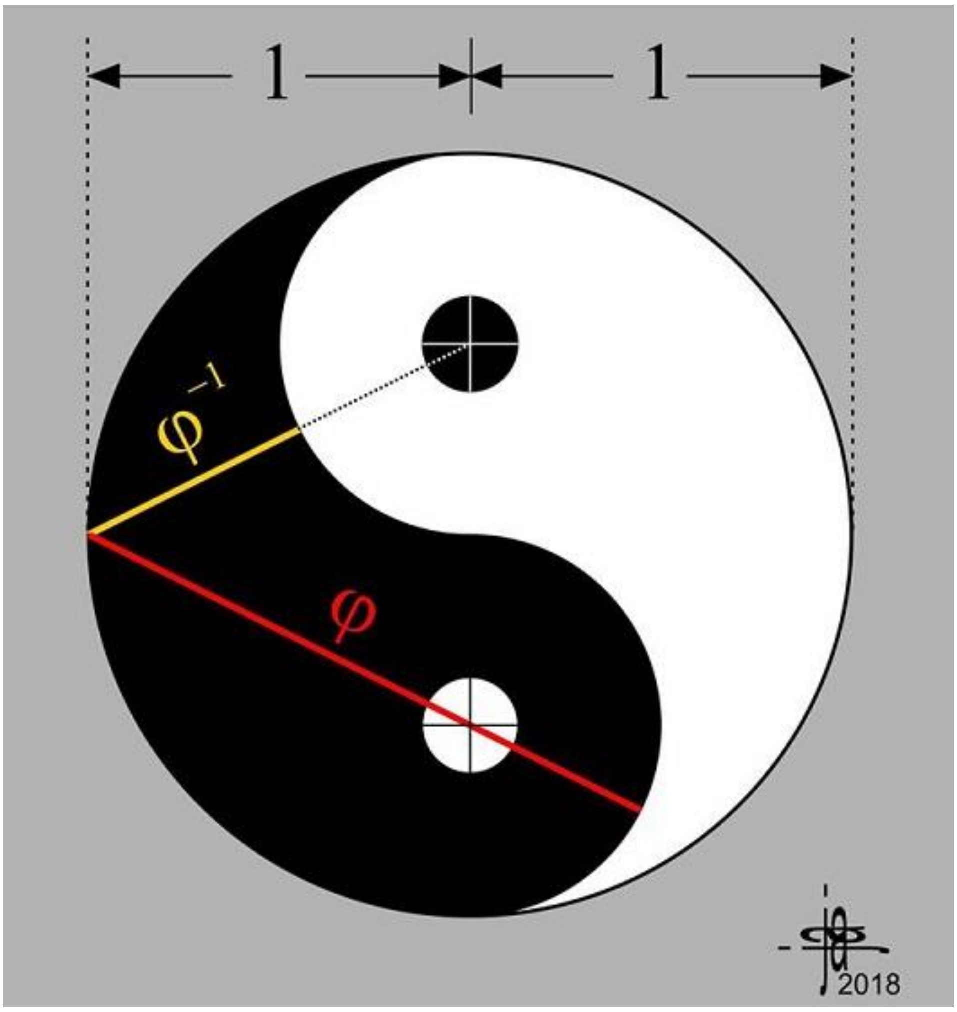

C_2
C_1
a
b
C_3
C_4
L
x_1
x_3
x_5
x_4
x_2

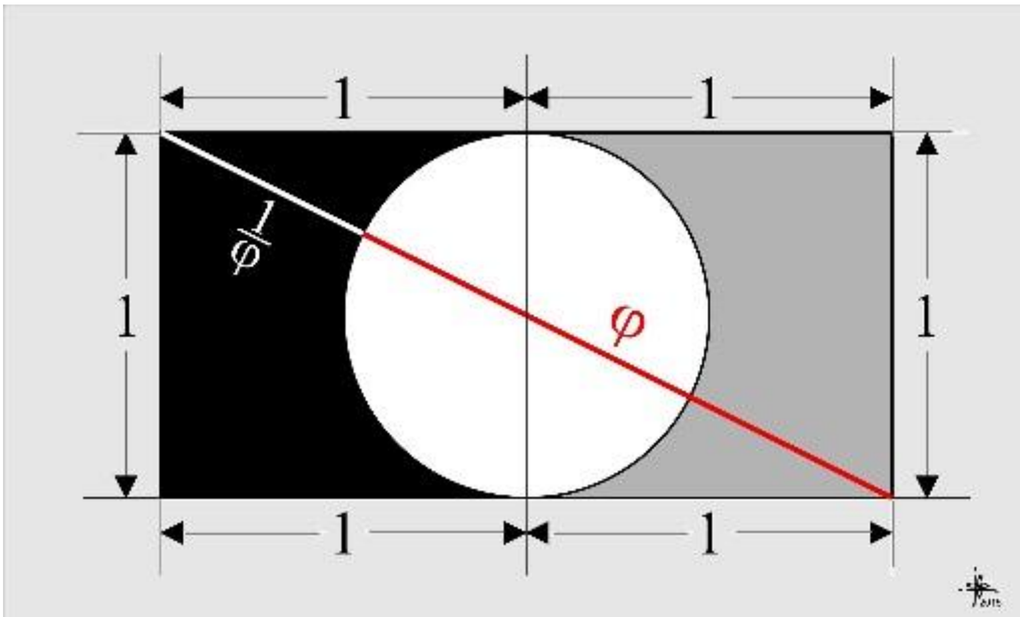



B
A
C
E
D
F
G

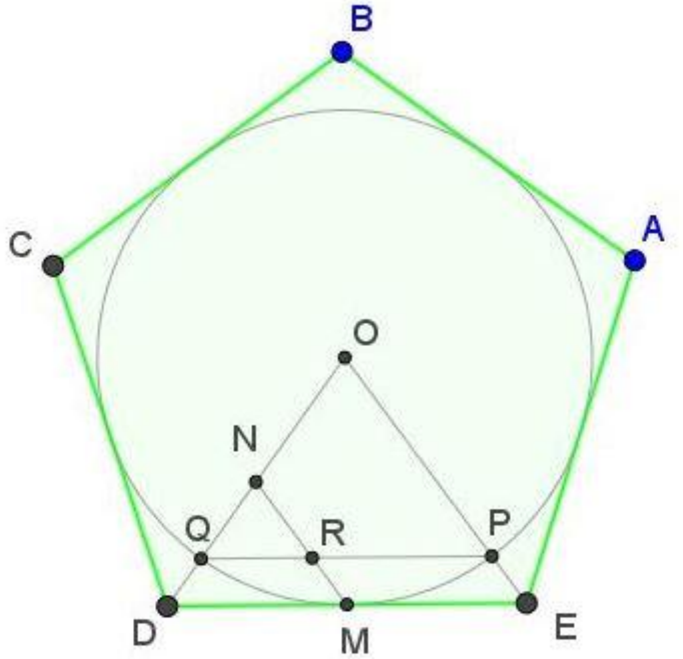

A
B
H
F
O
G
D
C
E
□ABCD, □BCEF : Square.
Prove that
$\frac{AB}{BH} = \phi$, the golden ratio.


A
M
D
N
E
P
B
C
ABC equailateral triangle
D,E touch points of incircle
M, midpoint of AE
MP//AB
PN/NM=Golden ratio
Reference. math-stackexchange.com

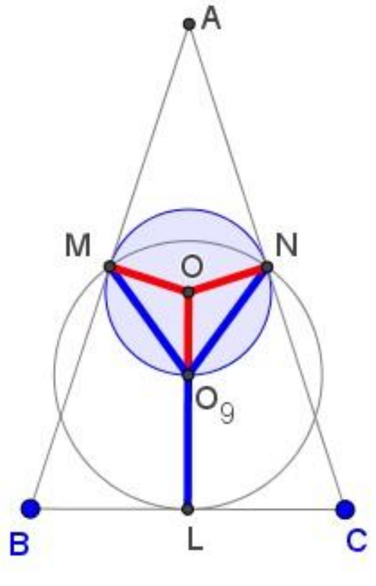

1
1
φ^{-1}
φ
2018

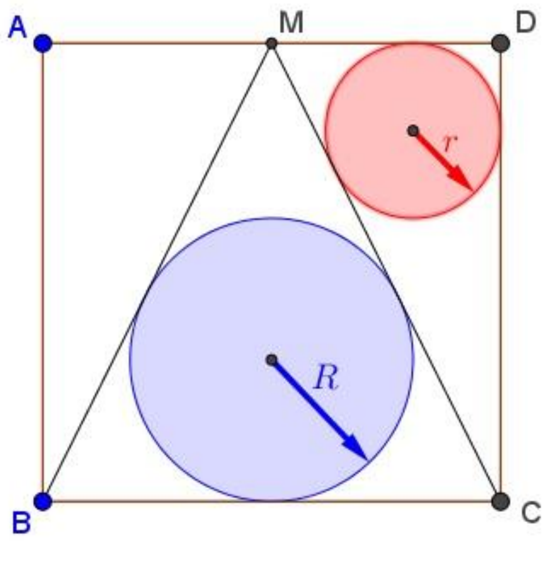

1
1
$\frac{1}{\varphi}$
φ
1
1
1
1

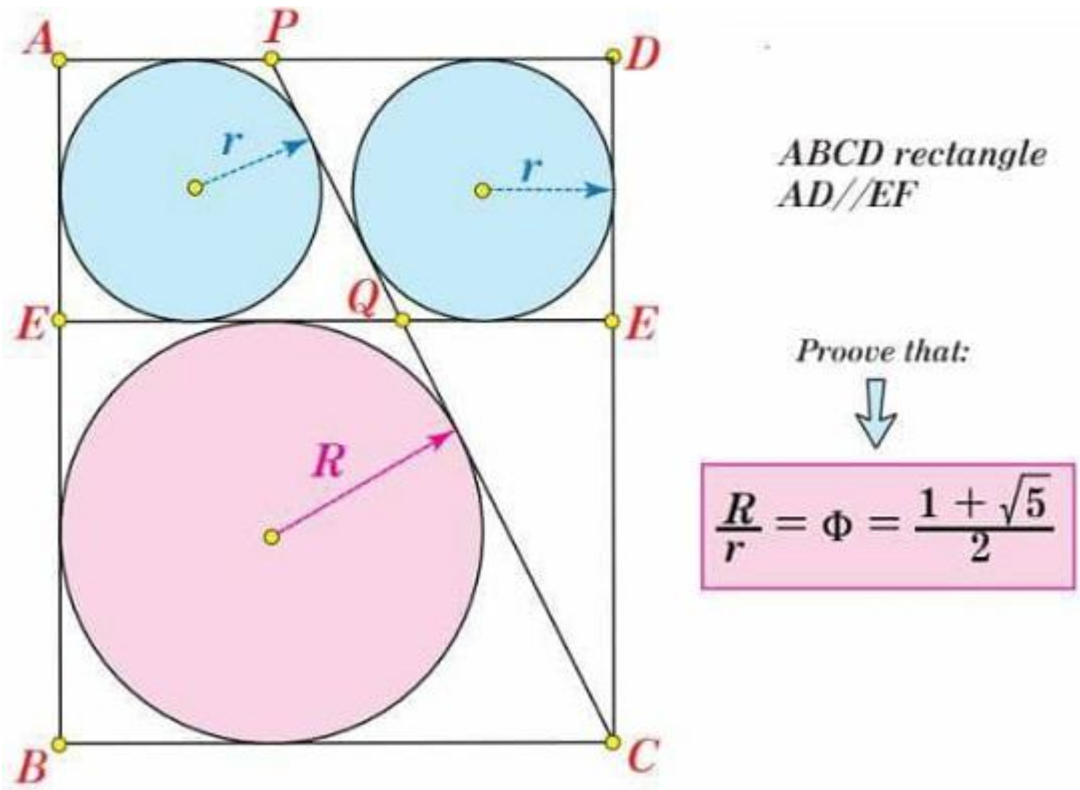
Let $ABCDE$ be a regular pentagon; M the midpoint of DE, N that of OD; P and Q the intersections of OE and OD with the incircle of the pentagon, respecively; R the intersection of PQ and MN.



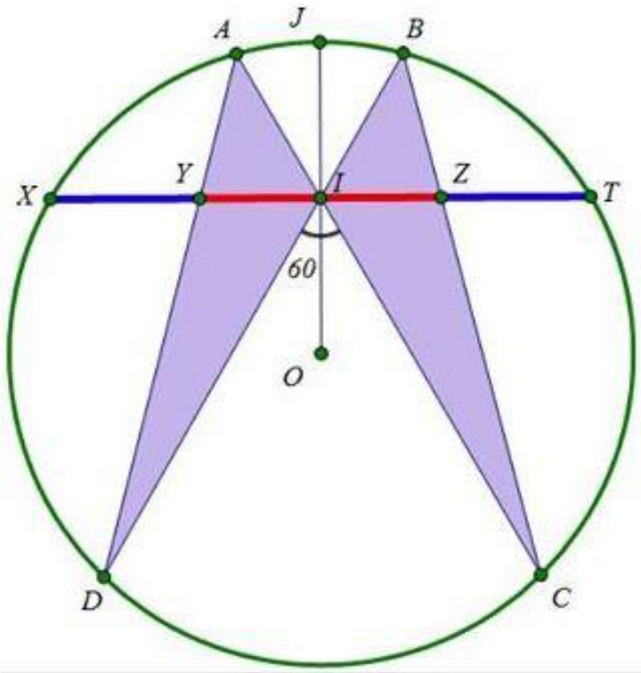
Then, $\dfrac{ON}{QN} = \dfrac{PR}{QR} = \varphi$, the **Golden Ratio.**







Let J lie on circle (O), with I the midpoint of OJ. Let C,D∈(O) make ΔICD equilateral. IC,ID cut (O) at A,B, respectively. Let XT be a chord through I perpendicular to OJ. AD,BC cut XT at Y,Z, respectively. Then, YZ/ZT = YT/YZ = φ.



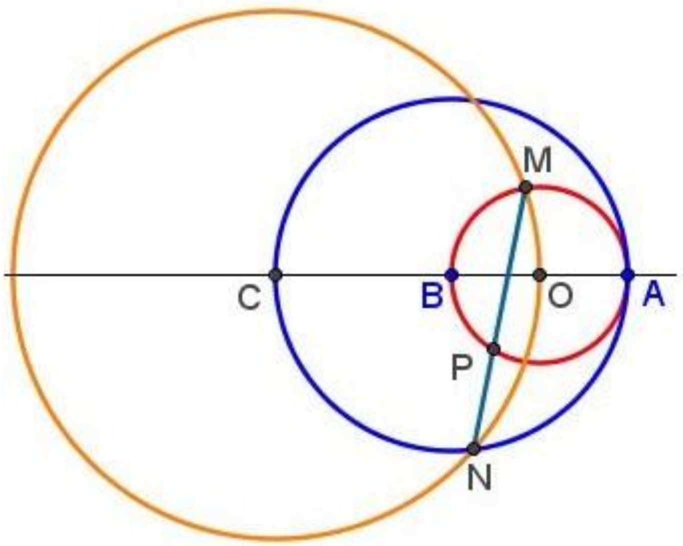
1. Draw line L and choose point O on that line.

2. Draw circle (O) of a random radius and mark the points of intersection of (O) with L, say A and B.

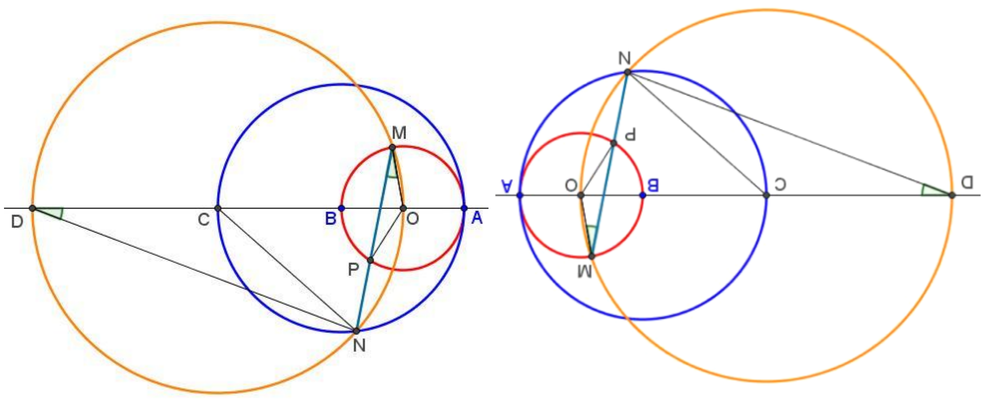
3. Draw circle $B(A)$ centered at B and passing through A. Let C be the second intersection of $B(A)$ with L.

4. Draw circle $C(O)$ centered at C and passing through O. Let it intersect $B(A)$ at N below L and (O) at M above L.

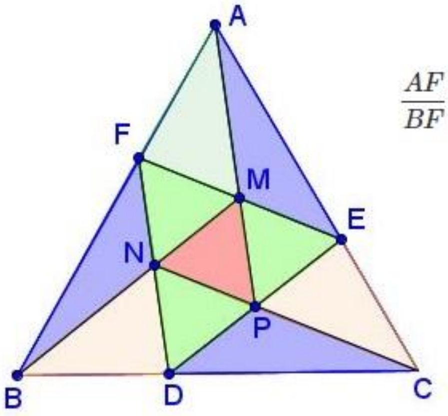
5. Join MN and let P be the intersection of MN and (O).



P divides MN in the golden ratio.



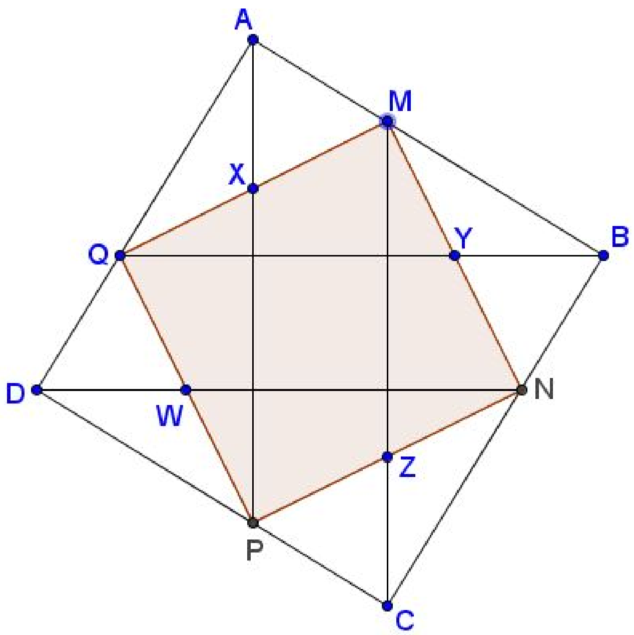
Equilateral ΔDEF is inscribed into equilateral ΔABC so that its extended midlines MN, NP, MP pass through the vertices of ΔABC, as shown:



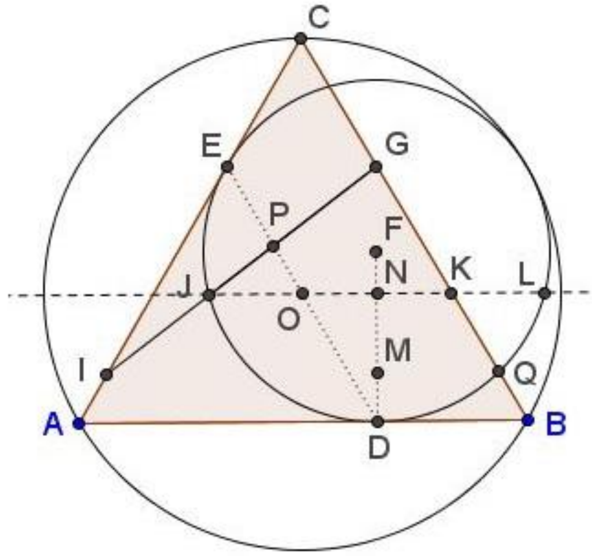
$$\frac{AF}{BF} = \frac{MN}{BN} = \frac{FN}{AM} = \frac{BF}{AB},$$

Then, say, the vertices of the inner triangle divide the sides of the outer triangle in the Golden Ratio, e.g., $\frac{BF}{AF} = \phi$.

Square $MNPQ$ is inscribed into square $ABCD$ so that the lines joining their vertices intersect the sides of $MNPQ$ at the midpoints X, Y, Z, W, as shown:



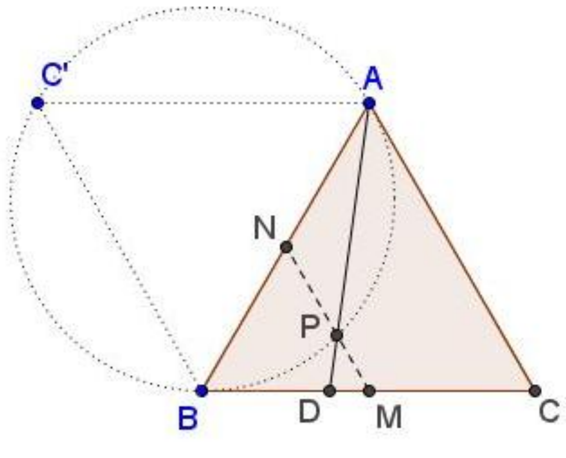
Prove that the vertices of $MNPQ$ divide the sides of $ABCD$ in the Golden Ratio, e.g. $\frac{BM}{AM} = \phi$.



(F is the center of the circle. The significance of other points is clear from the diagram.)

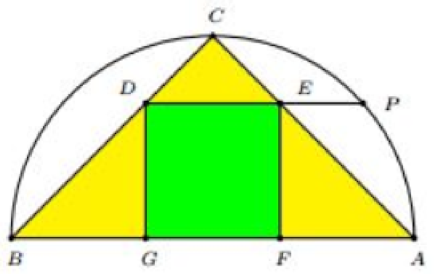
$$\frac{GP}{JP} = \frac{GJ}{IJ} = \frac{KO}{JO} = \frac{GK}{KQ} = \frac{KO}{KL} = \frac{GQ}{CG} = \frac{KQ}{BQ} = \phi.$$

Given two equilateral triangles ABC and ABC', M the midpoint of BC, N the midpoint of AB; P the intersection of MN with the circumcircle (ABC'); AP crosses BC in D.

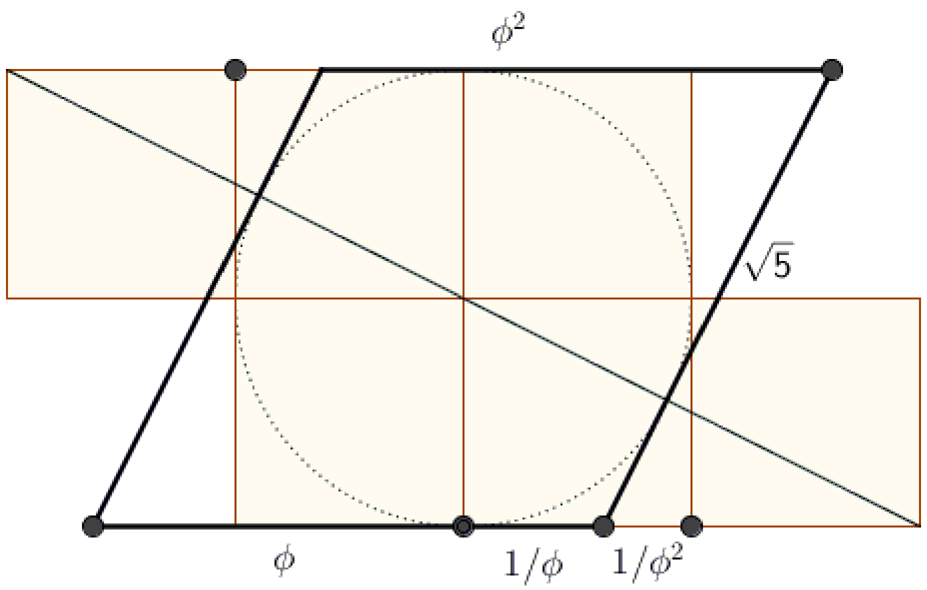


Then $CD/BD = \phi$, the Golden Section.

Given a right isosceles triangle ABC and its circumcircle, inscribed a square $DEFG$ with a side FG along the hypotenuse AB. Let the side DE extended beyond E intersect the circumcircle at P.



Then E divides DP in the Golden Ratio.



The Golden Number is a mathematical definition of a proportional function which all of nature obeys, whether it be a mollusk shell, the leaves of plants, the proportions of the animal body, the human skeleton, or the ages of growth in man and it turns out to be the key to understanding how nature designs and is a part of the same ubiquitous music of the spheres that builds harmony into atoms, molecules, crystals, shells, suns, galaxies and makes the Universe sing.

IV. Conclusion

Don't let all the math get you down. In design, the Golden Ratio boils down to the aesthetics creating and appreciating a sense of beauty through harmonic proportion. When applied to design, the Golden Ratio provides a sense of artistry; an X-factor; a certain *je ne sais quoi*. This harmony and proportion have been recognized for thousands of centuries. In fact, our brains are seemingly hard-wired to prefer objects and images that use the Golden Ratio. It's almost a subconscious attraction and even tiny tweaks that make an image truer to the Golden Ratio have a large impact on our brains.

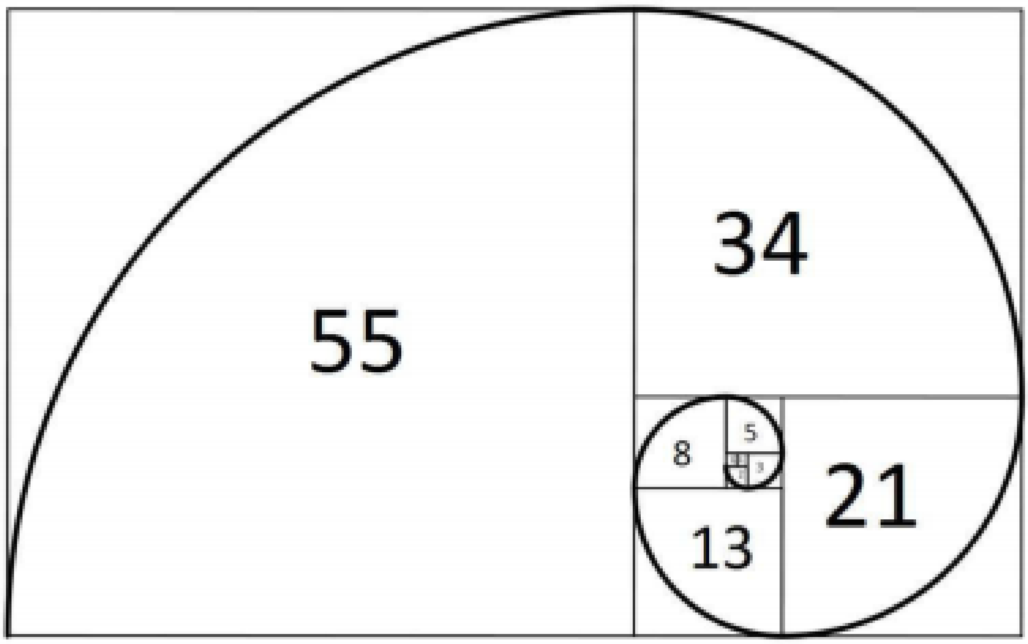


Figure: Fibonacci Spiral Constructed Using Concepts of Golden Ratio.

Acknowledgement

I would like to express my special thanks of gratitude towards my loving wife TAMANNA TABASSUM who inspired me to write this article.

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