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The position of the contact area, the contact stress distribution and the propagation of the circular surface waves during the collision between a spherical body and a semi-space surface

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Abstract: *The radius of the contact area has been found by consideration of the contact geometry. Two variants of the position of the contact area at impact between a spherical body and a semi-space surface have been examined in this paper. The differential equations of the displacement (the movement) of the centre of mass of the body have been given. The distribution of deformation velocities on the semi-space surface, and the size and velocity of the circular surface waves have been found. The occurrence of micro-impact waves in the initial stage of impact have been considered. The equation for the velocity of the front of the micro-impact wave has been obtained. Also, here, the problems of distribution of the stresses on the surface of the contact area and the definition of the normal contact stresses of elasticity and viscosity have been considered. It was shown that the general viscoelastic stress is the complex stress. An experimental technique for definition of the dynamic viscoelastic properties of the materials of the contacting surfaces at impact has been suggested. A comparison of two variants of the position of the contact area at impact has been given. It has been shown here in discussion that, using theories of the static (elastic, viscoelastic, elastoplastic) contacts and also so-called quasi-static contacts can be very problematic in solving problems of contact dynamics, and particularly at impact!*

Key words*: collision, impact, methods, specific forces, differential forces, viscoelasticity, dynamics, parameters viscoelasticity, specific viscoelastic forces, dynamic modules, energy, work, restitution, compression, contact geometry, dissipative energy and work, parabolic shape, movement differential equations, contact stress, stress distribution, velocities distribution, deformed volume, volume of deformation, circular surface waves, micro impact, oscillations, experimental techniques, experimental oscilloscopic curve, piezoelectric signal, time - temperature superposition, relaxation, physical relaxation, mechanical relaxation, relaxation of oscillations, retardation time, subatomic structures, viscoelastic behaviour, elastoplastic behaviour, plastic deformation, creep*

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I. Introduction

The dynamic contact between two smooth curvilinear surfaces can be considered as the process of their mutual collision at impact, which has two phases the compression (the loading) and the restitution (unloading). Also, in the case of the contact between two rough surfaces, we can assume that the contact of two asperities is similar to the impact between two smooth surfaces. Generally, it does not matter whether it is a sliding or rolling contact, but we can say that in the process of the contact, the volumes of deformation of the two contacting bodies are always involved in two phases of contact as at impact, that is, in the phase of compression and restitution. But, on the other hand, it is obvious that, a static or a quasi-static contact is not similar as at impact. The question arises: What indeed is the difference between the static, quasi-static solution of contact problems and the dynamic contact problems at impact? Here in this article, a more complete explanation of the difference has been presented.
Remark: First of all, we should understand that in static contact the sum of all forces between contacting bodies equals zero and there is no acceleration. Similarly, some scholars take that, in so-called quasi-static contact, if the applied forces vary slowly in time and with very slow acceleration, the fictitious force (also referred to as the inertia force) in the equations of motion can be ignored. Dynamic contact is totally distinctive from static or quasi-static behaviour! A dynamic system can be considered as being in an equilibrium, if a fictitious d'Alembert force, which equals the absolute value to the Newton force, is applied! But it is not a quasi-static behaviour between contacting bodies.

II. The geometry of the contact between two spherical surfaces

To determine the viscoelastic forces we have to know the equations or the expressions for $r = f(x)$, and for $h_x = f(x)$,(see Figure 1). For example, we can use the radius of a contact area $r = (Rx)^{1/2}$ according to the Hertz theory, but according to this theory, the area of contact is a flat surface and the depth of indentation (the depth of the contact surface) $h_x = 0$. But in reality, the area of contact usually is not a flat surface, but is a curvilinear surface. In Hertz's theoretical models, it has been taken that the contacting surfaces deform together without any sliding, but in reality, each surface deforms independently. Therefore, to find the radius of the contact area r in reality, let us to consider the geometry of the contact between two spherical surfaces, as depicted in the illustration in Fig.1.

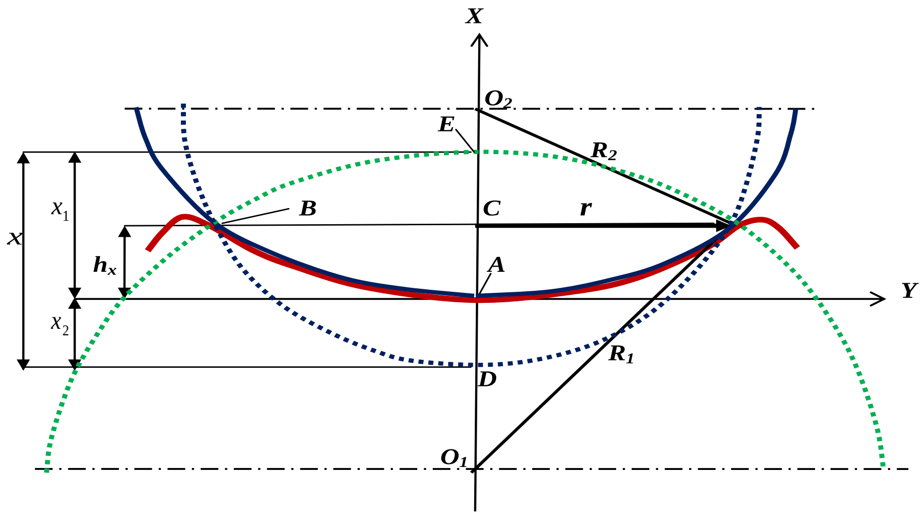

Fig. 1. Illustration of the contact between two spherical surfaces.

We know that a collision of freely moving bodies is a special state; it is the period of time when the colliding bodies are not affected by any external forces. It is not a compression of two bodies under the influence of an external force when only certain parts of the bodies in the contact zone are deformed. In the initial instant of time, during the freely moving collision of two bodies or two particles, the Newtonian force of inertia begins to act: $F_x = -m\ddot{x} = -(\sum_{i=1}^{n} m_i)\ddot{x}$, where m_i is the elementary mass of the body. If the initial speed of impact is less than the speed of sound inside the volume of deformation, all elementary masses of a body will be involved in the movement together, in the same time, and all the space of a body will be deformed in the same time as well. If a body is elastic or viscoelastic, the position of the centre of mass of the body relative to the initial position of the main axes of inertia of the body will not be changed and the magnitudes of the moments of inertia of the body will not change during the time of a collision, because if they were to change, the continuity of the environment inside the body would be broken. Also, it is obvious that, in the time of indentation of a hard surface into a soft surface, the contact surface takes a curvilinear shape, where the point ***B*** (see Fig.1,) is a special point where the deformations always equal zero, and the border of the area of contact always passes through this point ***B***. According to this statement, for example in the case of contact between two spherical bodies (see Fig.1), the distance O_2B between this point and the centre of curvature O_2 of the surface of the harder body will not be changed in the period of time of the contact. This distance always equals the radius of curvature R_2. Also, the distance O_1B between this point and the centre of curvature O_1 of the surface of the less hard body will not be changed during the period of time of contact, either. This distance always equals the radius of curvature R_1. Hence, we can see that $O_2B = O_2D = R_2$ and $O_1B = O_1E = R_1$, and also we can write that $O_1C + O_2C = (R_1 + R_2) - x$, and since as $O_1C = (R_1^2 - r^2)^{1/2}$and $O_2C = (R_2^2 - r^2)^{1/2}$, after a simple calculation, if we neglect the members of smallest order, we get the following equation for the radius of the contact area $r = f(x)$:

$$r^2 = 2Rx - x^2, \quad (1)$$

where $R = \frac{R_1 R_2}{R_1 + R_2}$ is the effective radius of contact curvature. Now equation (1) is not in a convenient form, therefore let us rewrites it as

$$r^2 = k_p^2 Rx, \quad (2)$$

where $k_p = \sqrt{2 - \frac{x}{R}}$ is the correlation coefficient. If a deformation is small, when $R >> x$, hence $k_p = \sqrt{2}$. Practically, for the solution of the contact problems of mechanics, the correlation coefficient can be found using iterations and consecutive approximations.

Now let us to define $\boldsymbol{h_x}$, which is the depth of the contact surface. The expression for the radius of contact area can be found also as follows (Fig.1):

$$r^2 = R_2^2 - (R_2^2 - (x_2 + h_x))^2 \tag{3}$$

After a simple geometric calculation, if we neglect by members the smallest order, we obtain the next equation for the radius of the contact area:

$$r^2 = 2R_2(x_2 + h_x) \tag{4}$$

Then, after comparing equations (1) and (4) we can state that

$$2R_2(x_2 + h_x) \approx 2Rx \tag{5}$$

Finally, since $x_2 = D_2 x$, where $D_2 = \frac{E_1'}{E_1' + E_2'}$ is the coefficient of deformation of the harder body, and E_1' is the dynamic elasticity modulus of a semi-space; E_2' is the dynamic elasticity modulus of a body, see [1,2,3], the formula for h_x can be written as follows

$$h_x = \left(\frac{R - D_2 R_2}{R_2}\right) x = k_h x, \tag{6}$$

where $k_h = \left(\frac{R - D_2 R_2}{R_2}\right)$ is the coefficient of the depth of the contact surface. Since in the case of contact between a spherical body and a semi-space, when $R_2 = R$ it follows that $k_h = (1 - D_2) = D_1$, and it is known that $x_1 = D_1 x$, see [1,2,3], where $D_1 = \frac{E_2'}{E_1' + E_2'}$ is the coefficients of deformation of the softer body, hence we get

$$h_x = x_1 = D_1 x \tag{7}$$

Also, using the MSF and MDSF methods, [1,2,3], the elastic force F_{cn} and viscous force F_{bn} can be found as $F_{cn} = 2\int x da$ and $F_{bn} = 2\eta' \int d\dot{x} \int da$. Also, since the diameter of contact area is defined by $a = 2r$, and we know the contact radius from (2), we can find the derivative for a by x, as $da = \frac{k_p R^{1/2}}{x^{1/2}} dx$, and taking into account that the effective dynamic viscosity $\eta' = \frac{E''}{\omega_x}$, then after integration we get, respectively:

$$F_{cn} = \frac{4}{3} k_p E' R^{1/2} x^{3/2} \tag{8}$$

and

$$F_{bn} = 4 k_p R^{1/2} \frac{E''}{\omega_x} \dot{x} x^{1/2} \tag{9}$$

where E'' is the effective dynamic modulus of viscosity, ω_x is the frequency of a damping oscillation.

III. Determination of the position of the contact area at impact between a spherical body and a semi-space

3.1 Varian 1.

It is known, from a large number of theories of contact mechanics, that, in static conditions of loading, the mutual approach between a spherical body and a semi-space occurs according to the scheme, as shown in Figure 2.a, when the contact area displaces relative to the initial point of contact at the size equals (x_1 - h_x). Therefore, let us assume here that, during the impact time, a similar approach between surfaces occurs, i.e. when the boundary of the contact area passing through point $\boldsymbol{B}$ it shifts relative to the surface of the semi-space, as shown in Figure 2.a. In this case, the volume of the semi-space V_h, which is in direct contact with the surface of the sphere, can be defined as

$$V_h = V_{d1} + \Delta V_T = V_{x1} - V_{c1} + \Delta V_T \tag{10}$$

Where, respectively: V_{c1} is the volume of compression of a semi-space, $V_{d1}$is the volume of deformation of a semi-space, which is in contact with a body; V_{x1} is the complete volume of deformation of a semi-space and ΔV_T is the increase in the volume of the half-space as a result of heating. The volume of semi-space, as the result of heating, is defined as $\Delta V_T = V_{x1}K_T\Delta T$, where K_T is the coefficient of volume thermal expansion, and ΔT is the amount of heating, which equals the difference of temperatures.
Further, the volume of compression of a semi-space can be found from the balance of the strain energy W_{x1} and the work A_{x1}, namely:

$$A_{c1} = W_{x1} = \int_0^{x_1} F_{cn}\, dx_1 = K_1' V_{c1} \tag{11}$$

where K_1' is the dynamic bulk modulus of elasticity of a semi-space and accordingly where F_{cn} is force of elasticity. Here we do not need to take into account the work of the dissipative force F_{bn}, as, during compression, it turns into heat.

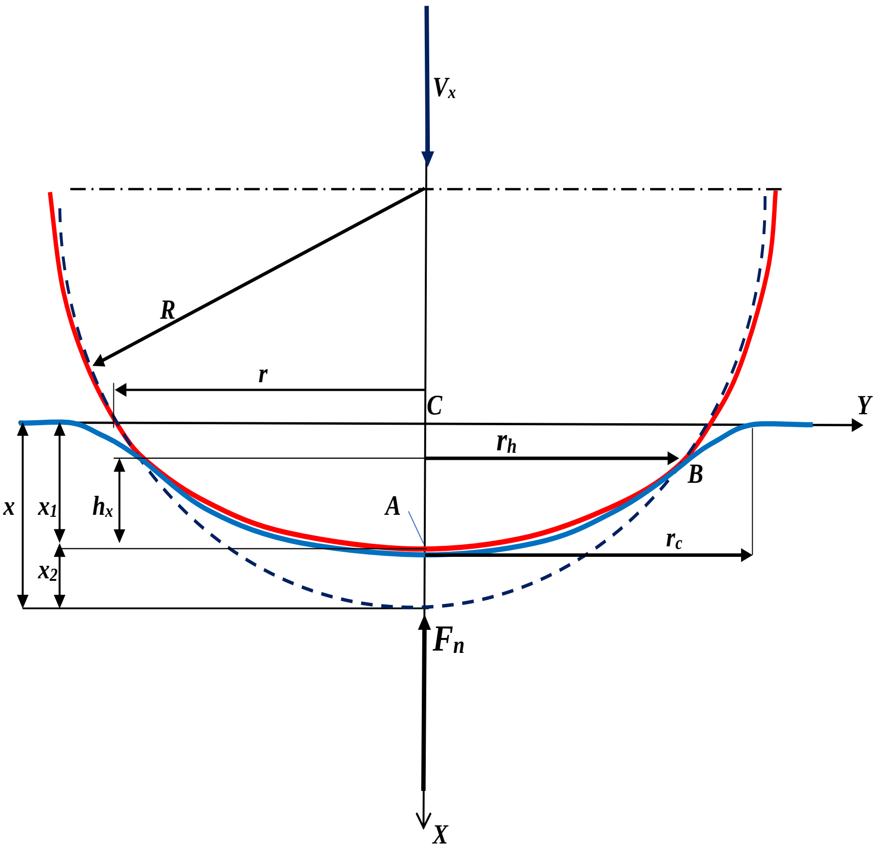

Fig. 2.a. The variant 1: Illustration of a viscoelastic indentation of a spherical body into a semi-space upon impact.

And since, as a result of compression the surface of the contact takes the parabolic shape (see Figure 2.a), it is obvious that

$$r_c^2 = k_c^2 R x_1 \tag{12}$$

and

$$r_h^2 = k_c^2 R h_x, \tag{13}$$

where k_c is the constant, and since, $x = \frac{x_1}{D_1}$, where $D_1 = \frac{E_2'}{E_1' + E_2'}$, we get

$$r_c^2 = k_c^2 R D_1 x \tag{14}$$

Also, since the equations for the differential forces are known [1,2,3], the expression for the normal differential elastic force in the case of circular contact can be written as $dF_{cn} = 2E'dxda$, and therefore

$$F_{cn} = 2E' \int dx \int da = 2E' \int x da \quad (15)$$

Since, in this case $da = 2dr_c$ and since, according to Eq. (14) $r_c = k_c R^{1/2} D_1^{1/2} x^{1/2}$, we can find the derivative for *a* by *x*

$$da = \frac{k_c R^{1/2} D_1^{1/2}}{x^{1/2}} dx \quad (16)$$

Then, after integrating (15) we get

$$F_{cn} = \frac{4}{3} k_c E' R^{1/2} D_1^{1/2} x^{3/2} \quad (16^*)$$

and since $x = \frac{x_1}{D_1}$, $E' = E_1' D_1$, it follows that $F_{cn} = \frac{4}{3} k_c E_1' R^{1/2} x_1^{3/2}$, and after integrating Eq. (11) it follows that

$$A_{c1} = \frac{4}{3} k_c E_1' R^{1/2} \int_0^{x_1} x_1^{3/2} dx_1 = \frac{8}{15} k_c E_1' R^{1/2} x_1^{5/2} \quad (17)$$

and taking in account Eq. (11), finally we get

$$V_{c1} = \frac{8}{15} k_c \frac{E_1'}{K_1'} R^{1/2} x_1^{5/2} \quad (18)$$

Note: Alternatively, the deformed volume of a semi-space can be found from the conditions of compression, as

$$V_{c1} = \frac{1}{K_1'} V_{x1} \Delta P, \quad (19)$$

where $\Delta P = P_t - P_0 = \frac{A_{x1}}{V_{x1}}$ and where P_t, P_0 are the current and the initial means pressures inside the volume of deformation. Further, since the area of the complete volume of deformation on the surface of the semi-space $S_x = \pi r_c^2$, we can determine the complete volume of deformation of a semi-space, as

$$V_{x1} = \int_0^{x_1} S_x \, dx_1 = \frac{\pi}{2} k_c^2 R x_1^2 \quad (20)$$

If the frequency of impacts on a surface is low, then the effect of heating can be neglected and then we get

$$V_h = V_{d1} = V_{x1} - V_{c1} \quad (21)$$

Substituting Eq. (20) and Eq. (18) into Eq. (21) gives

$$V_h = \frac{\pi}{2} k_c^2 R x_1^2 - \frac{8}{15} k_c \frac{E_1'}{K_1'} R^{1/2} x_1^{5/2} \quad (22)$$

Alternatively, (See Fig. 2.a), this volume is defined as

$$V_h = \int_0^{h_x} S_{hx} \, dh_x, \quad (23)$$

where the contact area $S_{hx} = \pi r_h^2$ and therefore

$$V_h = \pi k_c^2 R \int_0^{h_x} h_x \, dh_x = \pi k_c^2 \frac{R}{2} h_x^2 \quad (24)$$

And so, according to Equations (22) and (24), and taking into account that $\frac{E_1'}{K_1'} = 3(1 - 2\nu_1)$, where ν_1 is a coefficient of Poisson of a semi-space, we finally get

$$h_x = x_1 \left[1 - \frac{16}{5\pi k_c}\left(\frac{x_1}{R}\right)^{1/2} \times (1-2\nu_1)\right]^{1/2} \tag{25}$$

This expression can be written in the simplified form, as

$$h_x = x_1[1-\alpha_h]^{1/2}, \tag{26}$$

where

$$\alpha_h = \frac{16}{5\pi k_c}\left(\frac{x_1}{R}\right)^{1/2} \times (1-2\nu_1) \tag{27}$$

Also, alternatively, since we know that $r^2 = k_p^2 R x$, it follows, see Figure 2.a, that

$$r_h^2 = k_p^2 R(h_x + x_2) \tag{28}$$

A comparison of Eq. (13) and Eq. (28) gives

$$k_c^2 = k_p^2(1+\frac{x_2}{h_x}) \tag{29}$$

Here we are seeing that in the case where the body is much tougher than the semi-space and the deformation of the body is small compared to the deformation of a semi-space, we can take $k_c = k_p$. We also see that if the Poisson coefficient is equal to or close in magnitude to 0.5, it follows that $\alpha_h = 0$, accordingly, $h_x = x_1$. Or let, for example, the Poisson coefficient $\nu_1 = 0.4$, $D_1 = 0.8$, and the ratio $x/R = x_1/D_1R = 0.1$ (this is a fairly large deformation), and also taking into account that $k_c \approx k_p = \sqrt{2-\frac{x}{R}}$, we get $\alpha_h = 0.04186 \approx 0.042$! Hence, even in this case $h_x = 0.952x_1$. So, for this option and for bodies that are not very compressible we can set $h_x = x_1$.

3.2 Variant 2.

Figure 2.b illustrates the dynamic viscoelastic contact at impact of a spherical body and a semi-space.

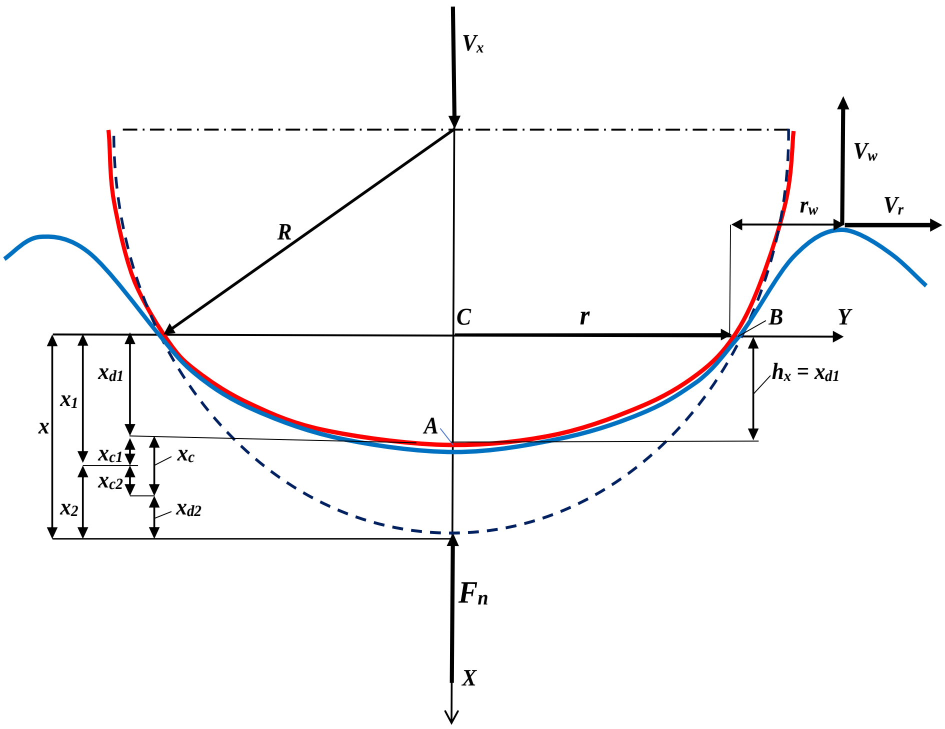

Fig. 2.b. The variant 2: Illustration of the picture of a viscoelastic indentation of a spherical body into a semi-space upon impact.

If in the Variant 1, the contact border is shifted, then in variant 2, the border of the contact area passing through point ***B*** is always located on the initial surface of the semi-plane, as in the case of dynamic contact between two spheres, given above in Figure 1. In this case, complete deformations are defined accordingly as: $x_1 = x_{c1} + x_{d1}$ and $x_2 = x_{c2} + x_{d2}$, where x_{c1} is the compression of a semi-space, x_{d1} is the elastic deformation of a semi-space, x_{c2} is the compression of a body, x_{d2} is the elastic deformation of a body. It must also be understood here, that the deformation of the compression of a body and a semi-space (see Figure 2.b) does not involve the penetration of the body into the semi-space, because $h_x = x_{d1}$, but these factors must be taken into account in moving the center of mass of the body, as $x = x_1 + x_2 = x_{c1} + x_{c2} + x_{d1} + x_{d2}$. Since the contact radius is general, see Figure 2.b, we can write that

$$r^2 = k_p^2 Rx = k_h^2 Rh_x = k_{x1}^2 Rx_1 \tag{30}$$

Hence

$$k_h^2 = \frac{x}{h_x} k_p^2 \tag{31}$$

and

$$k_{x1}^2 = \frac{x}{x_1} k_p^2 \tag{32}$$

Further, since the area of the complete volume of deformation on the surface of a semi-space $S_x = \pi r^2 = \pi k_{x1}^2 Rx_1$, respectively we get

$$V_{x1} = \int_0^{x_1} S_x \, dx_1 = \frac{\pi}{2} k_{x1}^2 Rx_1^2 \tag{33}$$

Further, taking into account that in this variant $F_{cn} = \frac{4}{3} k_p E' R^{1/2} x^{3/2}$ and since $x = \frac{x_1}{D_1}$, $E' = E_1' D_1$, and hence $F_{cn} = \frac{4}{3} k_p E' R^{1/2} \frac{x_1^{3/2}}{D_1^{1/2}}$, and then the integrating Eq. (11) gives

$$A_{c1} = \frac{4}{3} k_p E_1' \frac{R_1^{1/2}}{D_1^{1/2}} \int_0^{x_1} x_1^{3/2} dx_1 = \frac{8}{15} k_p \frac{R^{1/2}}{D_1^{1/2}} x_1^{5/2} \tag{34}$$

Now, the volume of compression of a semi-space can be found in a similar way, as has already been done using the expression in the equation (11), namely as

$$V_{c1} = \frac{8}{15} k_p \frac{E_1'}{K_1'} \frac{R^{1/2}}{D_1^{1/2}} x_1^{5/2} \tag{35}$$

Substituting Eq. (35) and Eq. (33) into Eq. (21) gives

$$V_h = \frac{\pi}{2} k_{x1}^2 Rx_1^2 - \frac{8}{15} k_p \frac{E_1'}{K_1'} \frac{R^{1/2}}{D_1^{1/2}} x_1^{5/2} \tag{36}$$

Alternatively, taking into account that in the Variant 2, $S_x = \pi r^2 = \pi k_h^2 Rx_1$, (see Figure 2.b) the volume V_h can be found in same way as in Eq. (23),

$$V_h = \pi k_h^2 R \int_0^{h_x} h_x \, dh_x = \pi k_h^2 \frac{R}{2} h_x^2 \tag{37}$$

Finally, in according with Eqs. (36), (37) and Eqs. (31), (32), since $\frac{E_1'}{K_1'} = 3(1 - 2\nu_1)$, we get

$$h_x = x_1 \left[1 - \frac{16 D_1^{1/2}}{5\pi k_p} \left(\frac{x_1}{R} \right)^{1/2} \times (1 - 2\nu_1) \right] \tag{38}$$

This expression can also be written in the simplified form:

$$h_x = x_1[1 - \alpha_h] \tag{39}$$

where

$$\alpha_h = \frac{16D_1^{1/2}}{5\pi k_p}\left(\frac{x_1}{R}\right)^{1/2} \times (1 - 2\nu_1) \tag{40}$$

Here, we also see that if the Poisson coefficient ν_1 equals to or closes to 0.5, it follows $\alpha_h = 0$ and respectively $h_x = x_1 = D_1 x$

Remark: If, for example, we take in Variant 1: $\nu_1 = 0.4$, $D_1 = 0.8$, the attitude $x/R = x_1/D_1R = 0.1$ (this is enough big deformation), and as well we consider that, $k_p = \sqrt{2 - \frac{x}{R}}$, we receive $\alpha_h = 0.03739 \approx 0.047$! Hence, in this case $h_x = 0.963\, x_1$. Thus, in Variant 1, for materials that are not very compressible we can set $h_x = x_1$.

3.3. Distribution of deformation velocities on the semi-space surface

According to Eq.2 the deformation (displacement) of the contact surface along the axis X has a parabolic shape $x = \frac{1}{k_p^2 R} r^2$, and taking into account that $r^2 = k_p^2 Rx$, and $r_i^2 = k_p^2 R(x - x_i)$ it follows that the velocity distribution function will also have a parabolic relationship, namely

$$V_{xi} = \dot{x}_i = \dot{x} f_x, \tag{41}$$

where

$$f_x = \left(\frac{r^2 - r_i^2}{r^2}\right) = \left(\frac{x_i}{x}\right) \tag{41*}$$

and where r_i is a current radius of the contact area. Also it can be seen that in the centre of contact the following is always true $x = x_i, f_x = \left(\frac{x_i}{x}\right) = 1$, but on the border of the contact area the following is always true $x_i = 0$, $f_x = \left(\frac{x_i}{x}\right) = 0$. Hence, the normal velocity on the contact area along the ***X*** axis, in the center of contact, equals $\dot{x}$, but it always equals zero on the border of the contact area! Similarly, we can find the tangential velocity of displacement of the contact surface along the ***Y*** axis, as

$$V_{ri} = \dot{r}_i f_r\ , \tag{42}$$

where

$$f_r = \left(\frac{r_i^2}{r^2}\right) = \left(\frac{x - x_i}{x}\right) \tag{42*}$$

It's obvious here, that in the centre of contact it's always true that $x = x_i, f_r = \left(\frac{x - x_i}{x}\right) = 0$, but on the border of the contact it's always true that $x_i = 0$, $f_r = \left(\frac{x - x_i}{x}\right) = 1$.

Thus, the tangential velocity of displacement in the center of the contact area always equals zero, but on the border of contact it always equals $\dot{r}_i$.

As we can see, the point of contact at $t = 0$ is an imaginary point, where the centre of contact and the border of contact area are coincident. Therefore, if we take $x = x_i$ we get $f_x = 1$ and $f_r = 0$, but if we take $x_i = 0$, we get $f_x = 0$ and $f_r = 1$. It is the paradox, because at the moment of time $t = 0$ it follows that $x = x_i = 0$.

IV. Surface waves

4.1 Size and velocity of the circular surface waves

The equation of the continuity of motion of a medium through the starting plane X of a semi-space can be written as

$$m_x(t) = m_e(t) \tag{43}$$

where $m_x(t)$ is the mass of compressing volume of deformation and $m_e(t)$ is the mass of expanding volume of deformation, see Figure 2.b.

Pulse conservation law must also be complied with

$$m_x V_x = m_e V_w \tag{44}$$

where $V_x(t)$ is the velocity of the compressing volume of deformation and $V_w(t)$ is the velocity of the expanding volume of deformation.
Only Variant 2 fully complies with these conditions. Obviously, Variant 2 can be used for dynamic contact on impact! Variant 1 can only be used for static loads if the contact area does not move at all!
Taking into account the compression of the volume of deformation, equation (43) of the continuity of a solid medium can be written in the following form:

$$\rho_c V_h = \rho_e V_e \tag{45}$$

where V_h is the compressing volume of deformation, V_e is the expanding volume of deformation $\rho_c = \rho_0\left(1+\frac{1}{K'}\right)$ is the density in the compressed state, ρ_0 is thedensity in the free state, K' is the dynamic bulk modulus.
The continuity equation (45) can also be written as follows:

$$\rho_c V_x S_x = \rho_e V_w S_w \tag{46}$$

where $S_x = \pi r^2$ is the surface of area of compression and S_w is the area of expanding, see Figure 2.b. But, since according to Eqs. (43) and (44), it follows that $V_x = V_w$, it also follows that,

$$\rho_c S_x = \rho_e S_w \tag{47}$$

Finally, we can write that,

$$S_w = a_k S_x, \tag{48}$$

where $a_k = \left(\frac{K'+1}{K'-1}\right)$. Also, it is obvious (see Figure 2.b) that, $S_w = \pi(r+2r_w)^2 - S_x$,

and then, we get this equation

$$4r_w^2 + 4rr_w - a_k r^2 = 0 \tag{49}$$

The solution gives us the next positive root

$$r_w = \frac{r}{2}\left(\sqrt{1+a_k} - 1\right) \tag{50}$$

If the material of a semi-space is not very compressible, when the dynamic bulk modulus $K' \gg 1$, we can take $a_k = 1$, and it follows in this case that,

$$r_w = \frac{r}{2}\left(\sqrt{2} - 1\right) \tag{51}$$

Now, according to Figure 2.b, the radius of the front of the impact circular surface wave can be shown as

$$r_{Aw} = r + r_w = \frac{r}{2}\left(\sqrt{2} + 1\right) \tag{52}$$

Since $r = (k_p x^{1/2} R^{1/2})$, we get

$$r_{Aw} = \frac{1}{2} k_p x^{1/2} R^{1/2}\left(\sqrt{2} + 1\right) \tag{53}$$

It is known [1,2,3] that the differential equation of the displacement (movement) of the centre of mass of a body can be written as follows:

$$m\ddot{x} + b_x\dot{x} + c_x x = 0, \tag{54}$$

where the expressions for the variable viscoelasticity parameters, see [1,2,3], can be written respectively as

$$b_x = \frac{4k_p E'' R^{1/2}}{\omega_x} x^{1/2}, \;\; c_x = \frac{4}{3} k_p E' R^{1/2} x^{1/2} \tag{55}$$

For practical application of the differential equation(54) with the variable viscoelasticity parameters, we can find their approximate solutions in the same manner as for the equations with the equivalent constant viscoelasticity parameters, if we choose the equivalent constant parameters B_x, C_x and B_y, C_y so that the work with the variable viscoelasticity parameters c_x, b_x will be equal to the work with the constant viscoelasticity parameters. And, according to the results obtained in the papers [1,2,3], the expression for the equivalent constant viscoelasticity parameters, respectively have been obtained as:

$$B_x = \frac{16E'' k_p R^{1/2}}{5\omega_x} x_m^{1/2}, \;\; C_x = \frac{16}{15} k_p E' R^{1/2} x_m^{1/2}, \tag{56}$$

where x_m is the maximum magnitude of the compression between a body and a semi-space (also it is the maximum displacement of the centre of mass of a body, which is equal to the maximum of mutual approach between a body and a semi-space).
Thus, equation (54) with variable parameters can be rewritten as per the equation with constant parameters as follows:

$$m\ddot{x} + B_x \dot{x} + C_x x = 0 \tag{57}$$

Equation (57) is the equation of the damped oscillations and the solution to this equation is known, as

$$x = \frac{v_{0x}}{\omega_x} e^{-\delta_x t} \sin(\omega_x t) \tag{58}$$

and respectively

$$\dot{x} = \frac{V_{0x}}{\omega_x} e^{-\delta_x t} [\omega_x \cos(\omega_x t) - \delta_x \sin(\omega_x t)] \tag{59}$$

Where: $\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2}$ is the angular (circular) frequency of the free damped oscillations; $\delta_x = \frac{B_x}{2m}$ is the normal damping factor; $\omega_{0x} = \sqrt{\frac{C_x}{m}}$ is the angular (circular) frequency of the natural harmonic oscillations along the X axis. It is clear that the period of time of the contact τ_x is equal to the semi-period of damped oscillations $T_x/2$ along the X axis.

$$\tau_x = \frac{T_x}{2} = \frac{\pi}{\omega_x} \tag{60}$$

Also, since $A_{xm} = W_{0x} = \frac{mV_{0x}^2}{2}$, $A_{xt} = W_{tx} = \frac{mV_{tx}^2}{2}$, where $V_{0x}$is the initial velocity, V_{tx} is the velocity of rebound, see [1,2,3], we can define the energetic coefficient of restitution e_x, which equals the square of the kinematic coefficient of restitution k_x (this will be referred to simply as the coefficient of restitution), as the ratio between work of restitution W_{tx} and work of compression W_{0x}:

$$e_x = k_x^2 = \frac{V_{tx}^2}{V_{0x}^2} = \left(\frac{\omega_x \tau_2 - 3tg\beta}{\omega_x \tau_1 + 3tg\beta}\right) \frac{\tau_1}{\tau_2}, \tag{61}$$

where $tg\beta = \frac{E''}{E'}$ is the tangent of mechanical losses.
Also, we can take that

$$x_m = \frac{/V_{0x}/}{2} \tau_1 = \frac{/V_{tx}/}{2} \tau_2 \tag{62}$$

where $\tau_x = \tau_1 + \tau_2$ is the period of time of the contact, τ_1 is the period of time of the compression, τ_2 is the period of time of the restitution. Thus, we get that

$$k_x = \frac{\tau_1}{\tau_2} \tag{63}$$

and using (61) and (63) we get that

$$tg\beta = \frac{\omega_x \tau_1}{3} \times \frac{1-k_x}{k_x} \tag{64}$$

Since $\tau_x = \tau_1 + \tau_2$ we get:

$$tg\beta = \frac{\pi}{3} \times \frac{(1-k_x)}{(1+k_x)} \tag{65}$$

The equation for the restitution coefficient we can now write as follows:

$$k_x = \frac{(\pi - 3tg\beta)}{(\pi + 3tg\beta)} \tag{66}$$

If *tgβ = 0* (hence $k_x = 1$) it is a totally elastic impact, but if *tgβ = π/2* (hence $k_x = 0$ and $x = 0$), it is a totally plastic impact. Both of these two cases are not possible in nature.
The expression for the maximum magnitude of the compression between a body and a semi-space has been obtained in [1,2,3], as

$$x_m = \left[\frac{15mV_{0x}^2}{16k_p E' R^{1/2}} k_x\right]^{2/5} \tag{67}$$

Also, it is clear that

$$x_m = \left[\frac{15mV_{0x}^2}{16k_p E' R^{1/2}} \times \frac{(\pi - 3tg\beta)}{(\pi + 3tg\beta)}\right]^{2/5} \tag{68}$$

Using Eq.(59) for the velocity, the duration time of the impact equals the period of time of the contact, which can be found now from the conditions $\dot{x} = V_{tx}$ and $t = \tau_x$, as

$$\tau_x = -\frac{ln\, k_x}{\delta_x}, \tag{69}$$

where

$$\delta_x = \frac{B_x}{2m} = \frac{8k_p E'' R^{1/2}}{5m\omega_x} x_m^{1/2} = \frac{8k_p E' tg\beta}{5\pi m} \tau_x R^{1/2} x_m^{1/2} \tag{70}$$

Also, the equation for the time of impact was given in [1,2,3], as

$$\tau_x^2 = -\frac{2(1+k_x)\, ln\, k_x}{V_{0x}^{2/5}(1-k_x)k_x^{1/5}} \times \left(\frac{5m}{8k_p E' R^{1/2}}\right)^{4/5} \tag{71}$$

Further, the equation of the circular viscoelastic surface waves, which arises on the surface of a semi-space of a solid body during the time of the excitation $t_w \geq \tau_1$ can be written as

$$\xi(y, t_w) = x_m e^{-\delta_x t_w} cos(\omega_x(t_w - \tau_1) - kr)\,, \tag{72}$$

where $k = \frac{2\pi}{\lambda_w}$ is the wave number and where λ_w is the wave length, and where ω_x is the frequency of the circular viscoelastic wave, which equals the angular (circular) frequency of the free damped oscillations.
Since the length of the wave can be found from Fig. 2.b, as $\lambda_w = 4r_{mw}$, and since the maximum of $r_{mw} = \frac{r_m}{2}(\sqrt{2} - 1)$ and $r_m = (k_p x_m^{1/2} R^{1/2})$, we get

$$\lambda_w = 2k_p x_m^{1/2} R^{1/2}(\sqrt{2} - 1) \tag{73}$$

Also, since, the phase velocity of this wave $v_w = \lambda\omega_x / 2\pi$, we finally get

$$v_w = \frac{\omega_x}{\pi} k_p x_m^{1/2} R^{1/2}(\sqrt{2} - 1) \tag{74}$$

The velocity of the circular surface waves is, in many instances, lower than the velocity of the acoustic elastic waves! For example, since it obvious that $\omega_x$has to be equal to the frequency of the acoustic elastic waves as well and since the velocity of an elastic transverse acoustic wave $V_s = \sqrt{\frac{G'}{\rho_0}}$, hence, the length of an elastic transverse acoustic wave can be calculated as $\lambda_w = 2\pi V_s/\omega_x$.

4.2 Micro impact surface waves

Since $r = (k_p x^{1/2} R^{1/2})$, the tangential velocity of growth of the contact area can, consequently, can be found as

$$V_{ri} = \dot{r}_i = \frac{k_p R^{1/2} \dot{x}}{2x^{1/2}} \left(\frac{x - x_i}{x}\right) \tag{75}$$

Also, it obvious (see Fig. 2.b.) that, the tangential velocity on the border of contact, where $x_i = 0$, is given by

$$V_r = \dot{r} = \frac{k_p R^{1/2} \dot{x}}{2x^{1/2}} \tag{76}$$

Also, the limit function $\lim\limits_{\substack{t \to 0^+ \\ x \to 0^+}} \frac{\dot{x}}{x^{1/2}} \to \infty$ Thus, Eq.(76) does not have a connotation in the case where $t=0, x=0$.

Also, the velocity of the front of a micro impact wave, when $t \ll \tau_1$, can, consequently, be found as

$$V_{Aw} = \dot{r}_{Aw} = \frac{k_p R^{1/2} \dot{x}}{4x^{1/2}} \left(\sqrt{2} + 1\right) \tag{77}$$

The function $V_{ri} = \dot{r}_i$ in Eq.(75) very quickly decreases and also, in the initial moment of the time $t = 0$, $V_{ri} = 0$, because $x - x_i = 0$, but, on the other hand, on the border of the contact, where $x_i = 0$, in a very short period of time $t = \Delta t$, and when displacement $x = \Delta x$ is also very small, the tangential velocity of growth of the contact area can reach a very high value! We can write, for the velocity on the border of contact, when $x = \Delta x$,

$$V_r = \dot{r} = \frac{k_p R^{1/2} \dot{x}_{t=\Delta t}}{2(\Delta x)^{1/2}} \tag{78}$$

We should understand, and it follows from Eq.(78) that, the velocity of the front of the micro-impact surface waves in the initial very short period of the time Δt can be higher than the velocity of an elastic transverse wave $V_s = \sqrt{\frac{G'}{\rho_0}}$, where $G'$is the dynamic shear modulus.

The occurrence of micro impact waves in the initial stage of impact contact on the surface of a semi-space will lead to a non-equilibrium state of the deformed medium and its rapid heating under the influence of thermodynamic impulse, because the work of the dissipative force will fully convert to heat!!! The impact time of the pulse will be equal to the deceleration time of the shock wave until the moment of impact $t=t_s$, $x=x_s$, when the propagation speed of the micro-impact wave becomes equal to the speed of sound, namely, when

$$V_{Aw} = V_s = \dot{r}_{Aw} = \frac{k_p R^{1/2} \dot{x}_s}{4x_s^{1/2}} \left(\sqrt{2} + 1\right) = \sqrt{\frac{G'}{\rho_0}} \tag{79}$$

Taking in account Eq. (58) and Eq. (59), we get

$$V_{Aw} = V_s = \dot{r}_{Aw} = \frac{k_p R^{1/2} \frac{V_{0x}}{\omega_x} e^{-\delta_x t_s} [\omega_x \cos(\omega_x t_s) - \delta_x \sin(\omega_x t_s)]}{4\left(\frac{v_{0x}}{\omega_x} e^{-\delta_x t_s} \sin(\omega_x t_s)\right)^{1/2}} \left(\sqrt{2} + 1\right) = \sqrt{\frac{G'}{\rho_0}} \tag{80}$$

The solution of this equation lets us define the time $t=t_s$. It is obvious that $t_s \ll \tau_1$

V. Contact stresses

5.1 Borders ofdistribution of the stresses on the surface of contact area

From Eq. (2) we can see that the surface of contact takes a parabolic shape $x = \frac{1}{k_p^2 R} r^2$, therefore, let us to assume that the radial distribution of contact stresses inside of this area changes analogically according to the parabolic function as

$$\sigma_i = \sigma_c \left(1 - \frac{r_i^2}{r^2}\right) = \sigma_c f_x \tag{81}$$

Further, since the square under this function and the square under the linear function of the mean stress σ_m in the contact area are equal, we get

$$\sigma_c \int_0^r \left(1 - \frac{r_i^2}{r^2}\right) dr_i = \sigma_m r \tag{82}$$

then after an integration it follows that

$$\sigma_c \left(r - \frac{1}{3} r\right) = \sigma_m r \tag{83}$$

and finally, the ratio between the maximum and mean stress in the contact zone can be found as

$$\sigma_c = \frac{3}{2} \sigma_m \tag{84}$$

Thus, we get the distribution of the normal stress inside the contact area as:

$$\sigma_i = \frac{3}{2} \sigma_m \left(1 - \frac{r_i^2}{r^2}\right) \tag{85}$$

Or since $f_x = \left(\frac{r^2 - r_i^2}{r^2}\right) = \left(\frac{x_i}{x}\right)$, as well, we can write that

$$\sigma_i = \frac{3}{2} \sigma_m \left(\frac{x_i}{x}\right) \tag{85*}$$

5.2 Normal contact stresses of elasticity

Since the mean stress in the contact area $\sigma_m = \frac{F_{cn}}{S_x}$, according to Eq. (84) the normal contact stress of elasticity in the centre of the contact area in the point ***A***, which equals the maximum of the contact stress, can be defined as

$$\sigma_{cn} = \frac{3F_{cn}}{2S_x} \tag{86}$$

and since $S_x = \pi r^2 = \pi k_p^2 R x$, we get

$$\sigma_{cn} = \frac{2E' x^{1/2}}{\pi k_p R^{1/2}} \tag{87}$$

Here, at the initial contact moment *t = 0*, the normal elastic contact stresses will be zero!

5.3 Normal contact stresses of viscosity

It is known that, in process of a viscoelastic hysteresis under a forced harmonic oscillation in during the first phase of unloading, in the point when deformation equals zero, the stress becomes delayed in the time from deformation by the value of the retardation time τ_{KE}. Also, it obvious analogy that, in process of the viscoelastic impact between bodies the retardation time in initial of moment of the time *(t=0)* should be equal zero, but in the moment of restitution τ_x it reaches its maximal value τ_{KE}! This retardation process can be described by the retardation function, as

$$f_{KE} = t \frac{\tau_{KE}}{\tau_x} \tag{87*}$$

Since, the normal mean stress of viscosity $\sigma_{mbn} = \frac{F_{bn}}{S_x}$, and since $S_x = \pi r^2$, $r^2 = k_p^2 R x$, and since F_{bn} is known from Eq. (9) respectively, the equation (85*) for the distribution of the normal stress of viscosity inside the contact area can be written, as

$$\sigma_{bni} = \frac{6E''}{\pi\omega_x k_p R^{1/2}} \times \frac{\dot{x}}{x^{1/2}}\left(\frac{x_i}{x}\right) \tag{88}$$

Here, taking in account the retardation function for stress in the expression (87*), the equations (58) and (59) can be respectively rewritten, as

$$x_\sigma = \frac{v_{0x}}{\omega_x} e^{-\delta_x t}\, sin(\omega_x t(1 - \tau_{KE}/\tau_x)) \tag{88a}$$

and respectively

$$\dot{x}_\sigma = \frac{V_{0x}}{\omega_x} e^{-\delta_x t}[\omega_x\, cos(\omega_x t(1 - \tau_{KE}/\tau_x)) - \delta_x\, sin(\omega_x t(1 - \tau_{KE}/\tau_x\,))] \tag{88b}$$

But, again we get a problem! If $x = x_i$, it follows that $f_x = \left(\frac{x_i}{x}\right) = 1$, but on the border of the contact area $x_i = 0$, and it follows that $f_x = \left(\frac{x_i}{x}\right) = 0$. Hence,the normal contact stress of viscosity in the centre of the contact area in the point ***A*** (see Fig.2b.) can be defined, as

$$\sigma_{bn} = \frac{6E''}{\pi\omega_x k_p R^{1/2}} \times \frac{\dot{x}_\sigma}{x_\sigma^{1/2}} \tag{89}$$

As we can see, the normal contact stress of viscosity at the center of the contact area in Eq. (89) at the moment of rebound $t = \tau_x$, $x = 0$ does not tend to infinity.

But, it seems that, the normal contact stress of viscosity in the center of the contact area in Eq. (89) in the initial moment of the time, $t = 0$, $x = 0$, tends towards infinity, because $\lim_{\substack{t\to 0^+ \\ x\to 0^+}} \frac{\dot{x}}{x^{1/2}} \to +\infty$. But, on the another hand, the centre of the contact area and the border of the contact area are coincident in the initial time $t = 0$, $\dot{x} = V_{0x}$. Thus, in the initial time, we get uncertainty $\sigma_{bn} = \infty 0$! But, this is not true, these are wrong points, because in these points the velocity$V_{0x}$is the real number and they cannot be divided by zero! Thus, thefunction in Eq.(89) does not have a connotation in the case $t = 0$, $x = 0$.

The point $t=0$, $x =0$ is the point of indeterminacy, since in the initial moment of the time the volume of the deformations and the area of contact equal zero, and here, in the zero zone, can arise only imaginary stresses. But, on the other hand, in a very short period of time Δt, when displacement $x = \Delta x$ is very small, the dissipative viscous contact stress in the microlayers of the contact surface can reach a very high value, which usually can be higher than the limit of the strength! Thus, this explains why usually in the initial micro time of contact under impact, there can be observed the damage, the recombination and the wear of the top layers of the surface of contact zone! The cause of these processes is the high value of the dissipative viscous stresses and the density of absorbed energy in the dissipative structures of the medium! Almost all the work of the dissipative viscous force in these conditions turns into heat!

Remark: If the deformed medium is not moving and it is in the static condition, it follows that the speed of deformation always equals zero, and therefore the viscous dissipative stress is not arising!

VI. Stresses under oscillations

To more properly understand the elastic and the viscous stresses, and to understand that they are have a completely different physical nature, it is simply necessary to consider the behaviour of viscoelastic media under an oscillation. Let the oscillation occur according to the sinusoidal harmonic law

$$\sigma^* = \sigma_0 e^{i\omega t},\, \varepsilon^* = \varepsilon_0 e^{i(\omega t - \beta)}\,, \tag{90}$$

where ω is the circular (angular) frequency of a forced oscillations, σ^* is the general viscoelastic stress, ε^* is the general viscoelastic deformation, σ_0 is the stress amplitude and ε_0 is the deformation amplitude. The relationship between them gives the expression for the complex modulus, as

$$\frac{\sigma^*}{\varepsilon^*} = E^* = \frac{\sigma_0}{\varepsilon_0} e^{i\beta} = \frac{\sigma_0}{\varepsilon_0}(\cos\beta + i\sin\beta) = \frac{\sigma_0}{\varepsilon_0}\cos\beta + \frac{\sigma_0}{\varepsilon_0} i\sin\beta, \tag{91}$$

where E^* is the complex modulus. Also here, we can designate that $E'=\frac{\sigma_0}{\varepsilon_0}\cos\beta$ is the dynamic modulus of elasticity (it is also named as the storage modulus), $E''=\frac{\sigma_0}{\varepsilon_0}\sin\beta$ is the dynamic modulus of viscosity (it is also named as the loss modulus),and hence

$$E^* = E' + iE'' \tag{92}$$

Also, it is known that

$$tg\beta = \frac{E''}{E'}, \tag{93}$$

where β is the angle of mechanical losses. Further, using Eqs. (92), (93) and since $\sigma^* = E^*\varepsilon^*$ we get

$$\sigma^* = \varepsilon_0 E'[\cos(\omega t-\beta) + \tan\beta\sin(\omega t-\beta)] + i\varepsilon_0 E''[\cot\beta\sin(\omega t-\beta) + \cos(\omega t-\beta)] \tag{94}$$

Thus, it follows that the general viscoelastic stress σ^* is the complex stress, and it can be expressed as the sum of the elastic stress σ' and the viscous stress σ'', respectively as

$$\sigma^* = \sigma' + i\sigma'', \tag{95}$$

where

$$\sigma' = \varepsilon_0 E'[\cos(\omega t-\beta) + \tan\beta\sin(\omega t-\beta)], \tag{96}$$

$$\sigma'' = \varepsilon_0 E''[\cot\beta\sin(\omega t-\beta) + \cos(\omega t-\beta)] \tag{97}$$

On the hand, since $E'=\frac{\sigma_0}{\varepsilon_0}\cos\beta$ and $E''=\frac{\sigma_0}{\varepsilon_0}\sin\beta$, we can express Eq.(96) and Eq. (97) in another form, as

$$\sigma' = \sigma_0[\cos\beta\cos(\omega t-\beta) + \sin\beta\sin(\omega t-\beta)], \tag{98}$$

$$\sigma'' = \sigma_0[\cos\beta\sin(\omega t-\beta) + \sin\beta\cos(\omega t-\beta)] \tag{99}$$

Here we can designate that

$$\varepsilon' = [\cos\beta\cos(\omega t-\beta) + \sin\beta\sin(\omega t-\beta)], \tag{100}$$

$$\varepsilon'' = [\cos\beta\sin(\omega t-\beta) + \sin\beta\cos(\omega t-\beta)] \tag{101}$$

Hence, we get

$$\sigma^* = \sigma_0(\varepsilon' + i\varepsilon'') = \sigma_0\varepsilon^* \tag{102}$$

Now, we can see that the viscous stress is the dissipative imaginary value, that arises in the process of impact inside dissipative structures of a deformed medium!
Since the impact between two bodies is a cyclic process, we can write that σ_{bn}=σ', $\sigma_{bn} = \sigma''$, and $\sigma^* = \sigma_n$, and using this statement, the general viscoelastic stresses can be found, analogically as it was in the paper [1], namely as

$$|\sigma_n| = \sqrt{\sigma_{cn}^2 + \sigma_{bn}^2} \tag{103}$$

VII. Experimental techniques for definition
of viscoelastic properties of materials of the contacting surfaces at impact.

We need to ask the question, what is the process of friction between contacting surfaces? The friction between moving surfaces is the dynamic contact at impact (a hit) between asperities, or if the contacting surfaces are smooth it is the cyclic jumps similar to frequently repeating dynamic impacts! See the "Tribocyclicity" section of the book [3]. Hence, for tribology problem solving, we should use the dynamic mechanical properties of materials of contacting surfaces, from testing! For example, we can use Dynamic Mechanical Analysis (DMA). But, as we already know, the dynamics behaviour of a medium at impact is so specific that, it is better to use the

experimental techniques for a definition of the viscoelastic parameters, which have been described in the book "Contact dynamics" [3].
For example, nobody is surprised when we use thermodynamics parameters such as thermal capacities, enthalpies, entropies and others thermodynamic properties of a medium from graphs or tables. It has been proposed that, for the definition of viscoelastic properties of materials of the contacting surfaces at impact, we can use the electro-mechanical device, the schema of which is depicted in Figure 3.

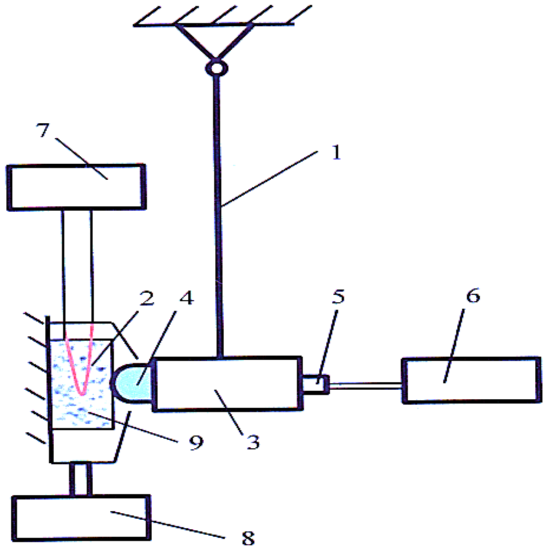

Fig. 3. The schema of the electro-mechanical device for the definition of the dynamic mechanical properties of materials at impact.

This device allows for the definition of the coefficient of restitution $\boldsymbol{k_x}$ and the time of impact $\boldsymbol{\tau_x}$depending on the temperature of the sample. The initial velocity of impact can be in the range $0 - 2.0\ m/s$. The device works as follows: the standard sample (9) is placed in a clip of the elasticity-meter The volume temperature of the sample is measured by means of the thermocouple (2) and digital milli-voltmeter (7) to within 0,05 ^{0}C with a milli-voltmeter resolution of 10^{-6}V. An electric signal is generated in the piezo-sensor (5), during the impact by the indenter (4) of the hammer (3) at the surface of the sample (9). The form of the signal (see Fig. 9) is registered by the oscilloscope (6). Elasticity on a rebound is defined mechanically by the size of the rebound of the hammer (3) and is fixed on a scale elasticity meter. A time of impact $\boldsymbol{\tau_x}$ is measured at the same time by the oscilloscope and an electronic frequency meter in a mode of measurement of current's impulse time, arising at impact in a piezo-element chain, and also in an electric chain the indenter-sample at contact of the hammer's indenter with the sample. Heating of the sample is carried out by hot air, cooling by streams of nitrogen using a heater and a cryogenic refrigerator (8). The thermocouple (2) is entered in the sample directly ahead of the beginning of tests. The thermocouple practically has no heat exchange with the outside environment. The temperature of the sample was registered at the moment of when the indenter (4) started to penetrate into the sample (9).
The example of the theoretical obtained curves for the viscoelastic forces are depicted in Figure 4. By comparison, Figure 5 shows a photo of the experimental oscilloscopic curve of the piezo-electric signal, which has been obtained as a result of the collision of a spherical steel indenter and a rubber sample. The power of the signal, which is generated by the piezo-detector placed inside the indenter, is directly proportional to the viscoelastic force. In the graphical comparison of these curves, which are presented in Figure 4 and Figure 5, we can see that the curves have very similar forms and hence this, once more, confirms that the theoretical solutions were made in the correct manner. Since, according to equations (62), (63) and (67), $\tau_x = \tau_1 + \tau_2$ we get the next expression, which allows us to find the instant dynamic modulus of elasticity

$$E' = \frac{(5.03)m}{k_p V_{0x}^{1/2} R^{1/2} \tau_x^{5/2}} \times \frac{(1+k_x)^{5/2}}{k_x^{3/2}} \tag{104}$$

And since the tangent of mechanical losses $tg\beta = \frac{E''}{E'} = \frac{\pi}{3} \times \frac{(1-k_x)}{(1+k_x)}$, see Eq.(65), we also get the expression for the instant dynamic modulus of viscosity as

$$E'' = \frac{\pi(5.03)m}{3k_p V_{0x}^{1/2} R^{1/2} \tau_x^{5/2}} \times (1 - k_x) \left(\frac{1+k_x}{k_x}\right)^{3/2} \quad (105)$$

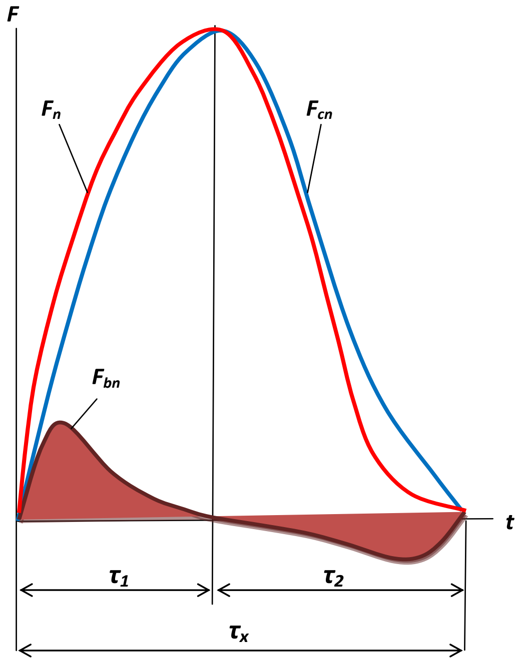

Fig.4. Graphical illustration of the results and the forms of curves obtained theoretically for viscoelastic forces.

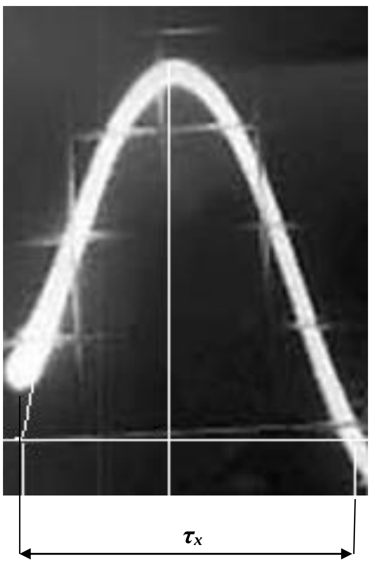

Fig. 5. Photo of the experimental oscilloscopic curve of the piezoelectric signal

Hence, using the obtained experimental values for the coefficient of restitution $\boldsymbol{k_x}$ and the time of impact $\boldsymbol{\tau_x}$ depending on the temperature of the sample respectively, we can calculate the temperature-velocity dependencies for the instant dynamic modules of elasticity and viscosity!

Also, the dynamic elasticity and viscosity modules for high velocities of collision can be found, if we follow the principles of the "Time-temperature superposition" according to the equation of "WLF" Williams - Landel - Ferry or Arrhenius [4, 5, 6,7]. First of all, we have to define experimentally the effect of temperature for the period of the contact time τ_x, and for the coefficient of restitution $\boldsymbol{k_x}$ at the fixed initial velocity of impact. For example, if we define these parameters for velocity at 2 m/s, then using the principles of the "Time-temperature superposition" we can determine their values for any velocities of interest, for example for velocity 100 m/c and for temperature 100^0C. After this, when $\boldsymbol{\tau_x}$ and $\boldsymbol{k_x}$ are known, we can find the value of ***tgβ*** and the dynamic modules $\boldsymbol{E''}$ and $\boldsymbol{E'}$.

According to the principle Temperature-Time (Velocity or Frequency) superposition, increasing the velocity or the frequency of loading is equivalent to decreasing the temperature of testing. For a description of the behaviour of amorphous polymers and elastomers the WLF equation is usually used, which has been proposed by Williams, Landel and Ferry

$$lg\, a_T = -\frac{C_1(T_{M.i}-T_g)}{C_2+(T_{M.i}-T_g)} \tag{106}$$

where $T_{M.i}$ denotes the temperature of mechanical glazing, T_g is the temperature of structural glazing, which is the constant of the material being examined, constant C_1=17.44, constant C_2=51.6 , and $lg\, a_T = lg\frac{V_g}{V_i}$, where V_g is the velocity of structural glazing when processes of the mechanical and the structural glazing coincidence, V_i denotes the initial velocity of impact under test. Index *i=1,2,3,..* for any $V_i \geq V_g$.
On the other hand, for description of behaviour of amorphous polymers and elastomers the Arrhenius equation is usually used

$$lg\, a_T = \frac{U_a}{2.3R_g}\left(\frac{1}{T_{M.i}} - \frac{1}{T_g}\right) \tag{107}$$

where U_a is the energy of activation of process of mechanical glazing, R_g is the universal gas constant.

VIII. Discussion and conclusion

The comparison of the two variants of the position of the contact area at impact, (Fig. 2a and Fig. 2b)leads us to the conclusion that, the Variant 1does not match the true picture of the dynamic contact in the process of impact, because in this variant the velocity of deformation along the *X* axis, at the contact boundary, will not be zero, and accordingly the contact stresses will not be zero at the boundary! Also, in Variant 1, the continuous conditions of the continuous medium and the impulse conservation law are not satisfied! Variant 1 can only be used for dead loads if the contact area does not move at all! Probably it is suitable for static deformation, when thevelocity $\dot{x} = 0$!

Therefore, the comparison of these two variants brings us to the conclusion that the Variant 2 is more realistic in the case of impact between curvilinear surfaces!

Now, you might say that all issues regarding axisymmetric static and quasi-static viscoelastic contact problems, including the relationship between indentation depth, contact radius and normal force, have already been solved in a rigorous way by Lee and Radok [8,9], Hunter [10], Graham [11], and Ting [12]et. al in the last century. They used a linear viscoelasticity theory in their research work. But, in those times, the classical theory of a viscoelastic contact mechanics still has not been properly developed, particularly for conditions of a dynamics contact between solids! They were pioneers in this area of research. But if someone invented a carriage and someone else invented a steam locomotive after, it can't be a ban on inventing a car or an airplane, can it? We can't stop the development of science, can we? Some people, if they do not see the profit for themselves, can make temporary barriers for innovative decisions, but they cannot completely stop them. It is not possible!

It is known that, Radok [8] and subsequently, Lee and Radok [9] have suggested to defined the time-dependent stresses and shear strains deformations for a quasi-static indentation between an axisymmetric indenter and a semi space for isotropic materials by using a linear viscoelasticity theory. They respectively have suggested to use the 'Boltzmann superposition principal', which is known, for example, in the form of an integral equation of Boltzmann-Volterra [13,14], (it is known elsewhere as a hereditary integral), as shown below

$$\varepsilon(t') = \int_0^t J(t-t')\frac{\partial\sigma(t')}{\partial t'}dt', \tag{108}$$

where *J(t)* is the function of creep (compliance) of a viscoelastic medium under shear and t' is the variable of integration.
Than, they have used so called “Elastic-Viscoelastic Correspondence Principle” to provide a resolution of the viscoelastic problems by using existing an elastic solutions. It is known that, according to the Hertzian contact solution, the connection between displacement *x* and normal elastic force can be written as

$$x^{3/2} = \frac{3}{4}\frac{F_n}{ER^{1/2}} \tag{109}$$

where *E* is known as the reduced Young’s modulus of elasticity since it is less than the combined elasticity of the two contacting solids $\frac{1}{E} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$, where E_1,E_2 are Young’s modulus of the contacting surfaces at the initial moment of the time, when $t = 0$ and ν_1, ν_2 are the Poisson’s coefficients of the contacting surfaces.
Lee and Radok have used Equation (109) for contact between a rigid sphere and viscoelastic semi-space of elastomer (rubber). Since a rigid body is many times harder than a soft semi-space of rubber, and since obviously $E_1 << E_2$ (where E_1 is modulus for elastomer (rubber) and E_2 is the modulus for material of rigid sphere), they took $\frac{1}{E} = \frac{1-\nu_1^2}{E_1}$, and since $E_1 = 2(1+\nu)G_1$, where G_1 is the shear modulus in the initial moment of the time, when $t = 0$, they deduced that

$$x^{3/2} = \frac{3}{8}\frac{F_n(1-\nu_1)}{G_1R^{1/2}} \tag{110}$$

Then, if considering the relation between the stress and the strain as a pure shear $\sigma = 2G_1\varepsilon$, and according to Eq. (110), then Eq.(108) transforms into the next expression

$$x(t)^{3/2} = \frac{3(1-\nu_1)F_n}{4R^{1/2}\sigma}\int_0^t J(t-t')\frac{\partial\sigma(t')}{\partial t'}dt' \tag{111}$$

Then after the recombination $F_n = \int_0^t \partial F_n(t')$ and $\int_0^t \partial\sigma(t') = \sigma$ in the Eq. (108), it follows consequently that

$$x(t)^{3/2} = \frac{3(1-\nu_1)}{4R^{1/2}}\int_0^t J(t-t')\frac{dF_n(t)}{dt'}dt' \tag{112}$$

Thus, we can see that, according to Radok and Lee theory, $\frac{F_n}{2G_1}$ in Eq. (110) can be replaced using the integral equation of Boltzmann- Volterra (108), which also is named as the integral operator, as follows:

$$\left[\frac{F_n}{2G_1}\right] \rightarrow \int_0^t J(t-t')\frac{dF_n(t)}{dt'}dt' \tag{113}$$

Remark: The solutions of Volterra equations can be found by using well known Laplace transforms [13,14].
Also, Radok and Lee [9] also pointed out that the contact solution in Eqs. (112) is only valid when the contact area increases monotonically. It means that velocity of deformation should be equal to a constant value! But it is not possible, indeed the velocity always varies during the contact process, because the contact forces and accelerations never could be constants! Therefore, and in the process of friction under colliding of asperities of the contacting surfaces, velocities and accelerations do not constants too! Hence, it is incorrect way to apply their theory in these cases!

Moreover, it is necessary to say here that, different choices for the load function $F_n(t)$ and the creep compliance *J(t)* can be made here, for example some research uses the Maxwell model, where $J(t) = \frac{1}{G_1} + \frac{t}{\eta_1}$; η_1 is the viscosity of the material of the soft contacting surface; and some use the Kelvin–Voigt model, where $J(t) = \frac{1}{G_1}[1 - exp(-t/\tau)]$; τ is the retardation time. These two models are the basic models, which are already used as elements for many complicated theoretical models in much research. The Kelvin–Voigt model is usually used for the description of the creep compliance, and the Maxwell model usually is used for the description the stresses relaxation.

But nevertheless, a lot of problems still exist: The first is that, a linear viscoelasticity can be applicable only when a material is submitted to deformations or stresses small enough (as in a quasi-static state), so that its rheological functions do not depend on the value of the deformation or stress, and the material response is in the linear zone of viscoelasticity. The second is that the viscoelastic force $F_n(t)$ and the creep compliance *J(t)* are not independent linear functions, but they are dependent on each other.

Also, Lee & Radok's method is based on the assumption that an elastic solution to a problem is instantaneously a viscoelastic solution too, and the compatibility and internal equilibrium equations are satisfied regarding the stress–strain–time relationships! But, the catch is that at impact or dynamic loading, boundaries of contact are moving so fast, that the boundary conditions may not be satisfied.

The problem is how to find the *J(t)*, specifically in relation to contact dynamics, because the creep compliance depends on the time of relaxation or the time of retardation, but on the other hand, the time of relaxation and the time of retardation depend on the times of loading and unloading. Also, at the initial moment of time of the dynamic viscoelastic contact, at impact $t = 0$, the normal and tangential contact forces are equal to zero, and therefore the contact stresses of elasticity and deformations do not increase instantly, but on the other hand, as we already know, in a very small period of time the dissipative stresses of viscosity can reach a very high magnitude, which can be higher than the limit value!

Furthermore, there are many problems in the application of this theory for high speeds and frequencies of dynamic contact, because, as a rule, the creep is a process of slow deformation under static load, which takes a long period of time, but if the time of loading is reduced the time of relaxation (or the retardation time) is reduced, too. Theoretical expressions and equations obtained for these are usually difficult to use in a practical application because they are very complicated, and there do not exist simple, convenient methods, which allow the finding of dynamic-mechanical parameters at impact, which are contained in these expressions and equations. Moreover, there is a problem in applying them in the case of a high-speed dynamic contact at impact, using the loss modulus and the storage modulus, obtained by the dynamic mechanical analysis (DMA) which uses excitation to create oscillation. And also, the Poisson's ratio (it is usually taken as $\nu = 0.5$ for rubber surfaces), which is obtained for the static or quasi-static loads but gives wrong results. Poisson's coefficient never reaches the value 0.5 and it is not a constant!

In the finding of solutions by using the linear viscoelastic stress-strain relationships, we have problems in the definition of the relaxation modulus in realistic temperature/time/speed/frequency conditions of dynamic viscoelastic contact.

We have to understand that, the time of a physical relaxation is the time of transition of a system of particles or elements in some volume of substance from the one equilibrium state into the next equilibrium state. It obvious that some molecular, sub-molecular or sub-atomic structures have their own times of relaxation, but the total relaxation time or the effective relaxation time is the time of relaxation, which matches the time when all relaxation processes inside a substance have completely finished. Therefore, this total time of relaxation has to be taken into account both when a volume of contact is an isotropic or an anisotropic medium.

But, on the other hand, the time period of the stress relaxation τ_{ME} in the Maxwell Model, is the time during which the stresses decrease in *e* times. It is not a physical relaxation, and in contact dynamics or under a cycling loading, it is calculated by the following formula

$$\tau_{ME} = \frac{1}{\omega \tan \beta} \tag{114}$$

The Maxwell Model can be used only for the description of the elastoplastic behaviour of contacting surfaces because, at the end of the phase of restitution, we always observe a plastic deformation! But others models, which include Maxwell Element, can be used for elastoplastic contact, but not for viscoelastic!

For the description of the viscoelastic impact between bodies we have to use the Kelvin- Voigt model! The retardation time τ_{KE} in this model is calculated as

$$\tau_{KE} = \frac{\tan \beta}{\omega_x} \tag{115}$$

Since $\omega_x = \frac{\pi}{\tau_x}$, the dependency between the retardation time and the time at impact can be expressed as

$$\tau_{KE} = \frac{\tau_x \tan \beta}{\pi} \tag{116}$$

As we can see, the retardation time is always many times less than the time of impact and it is always less than the time of the phase of compression τ_1. Here, it is important to reiterate that the period of the time of the compression is the period of excitation, when all the parameters of the medium are transferred into a specifically unstable state! But it is not an instantaneous process!

It is known that the time of relaxation of mechanical oscillations τ_ω is the time during of which the amplitude decreases in *e* times, and it can be calculated as

$$\tau_{\omega} = \frac{1}{\delta_x} \tag{117}$$

Thus, the relations between the retardation time and the time of relaxation of mechanical oscillations can be expressed as

$$\tau_{KE} = \tau_{\omega} \tan\beta \sqrt{\frac{1}{1+\omega_{0x}^2}} \tag{118}$$

In contact dynamics, it is very important to know the time period of the phase of viscoelastic compression between two surfaces, because at the point of maximum compression the velocity of mutual approach between surfaces equals zero, and part of the kinetic energy transforms into potential energy of deformations, and part of the kinetic energy dissipates as a result of internal friction and transforms into a heat. This is similar to the new equilibrium state. Thus, the time of the phase of compression (or time the maximum mutual approach) and the effective time of a physical relaxation in this phase have a common physical nature; and they are dependent on each other. Very often, when studying the phenomena of creep and stress relaxation, it is assumed that the initial stress or strain is applied suddenly, almost instantly! A natural question arises as to what will happen to the substance under conditions where force and, accordingly, strain or stress are applied suddenly or instantaneously. The result will be completely catastrophic, since acceleration in this case will tend towards infinity! But, such a development is not possible in reality! It is simply a theoretical trick! Moreover, it is such an obscure supposition that we can consider such a state to be a static or a so-called quasi-static state of a mechanical system or substance! Even if we were to load a sample of any material slowly, we would bring it into an unstable state and, of course, then we would still be able to observe the processes of a stress relaxation and a creep, like the transition of the sample material from a non-equilibrium into an equilibrium state. Here it is quite obvious that, when the loading of the sample occurs quickly, and the speed changes quickly, this is similar to what we observe at impact! Under such conditions, all thermodynamic and mechanical dynamic parameters in all solids and liquids, among other things, will be dependent on the time of exposure, and not only for rubbers, elastomers and polymers. All substances, without exception, depend on the time, because dissipative processes occur in them! But, at the same time, dissipative viscous stresses quickly disappear while the medium continues to move and the work of dissipative forces turns to heat! The forces of elasticity and viscosity act independently, because the viscous dissipation structures of the media and the bearing elastic structures of the media work independently too!

Also, it is usually taken by many researchers that, the cause of dissipation is only the internal friction in the materials and particularly inside elastomers (or rubbers) and it is primitively considered as pure shear between layers of material inside the block of elastomer or between layers of medium. Therefore, the modulus and viscosity at shear are used for the description of viscous deformations problems. But it is very difficult to agree with this statement that only pure shear is observed, because it is compression and because the contact stress is the stress of compression! And the dissipation also goes in the process of compression and as a result, the moving of molecular and sub-molecular structures take place! Also, the dissipation depends on many different causes, such as: the molecular or atomic structure of the medium, the entropy, the velocity of deformation and the temperature!

Also, as we know, in reality, the viscosity is a dissipative process of the changing of space-conformations between macro-molecules or between atoms (ions) in crystal grids. In a dependency of the type of deformation (extension or tension/compression, bulk compression and shear), there are six kinds of modulus and of viscosities in the descriptive dynamic mechanical properties of materials [4,5,6, 7]. For example, the main constitutive relations between them are known as:

$$E' = 3K'(1-2\nu) = 2G'(1+\nu)\,, \tag{119}$$

$$E'' = 3K''(1-2\nu) = 2G''(1+\nu) \tag{120}$$

and

$$E' = \omega\eta''_E\,,\ K' = \omega\eta''_K\,, G' = \omega\eta''_G\,, E'' = \omega\eta'_E\,, K'' = \omega\eta'_K, G'' = \omega\eta'_G \tag{121}$$

where E' is the dynamic elasticity modulus (or the storage modulus), which is equal to Young's modulus of elasticity in the Kelvin–Voigt model; $\eta'_E$is the dynamic viscosity; G' is effective dynamic elasticity modulus at the shear (or it is the effective storage modulus at the shear); η'_G is the effective dynamic viscosity at the shear. Then, according to Equations (119), (120) and (121), respectively we get

$$\eta'_E = 3\eta'_K(1-2\nu) = 2\eta'_G(1+\nu) \tag{122}$$

$$\eta_E'' = 3\eta_K''(1-2\nu) = 2\eta_G''(1+\nu) \tag{123}$$

Remark: If we exclude the Poisson coefficient, the resulting equation for the relationship between static elasticity constants is as follows $E = \frac{9GK}{G+3K}$, where K is the bulk modulus!

Also, what is indeed the correspondence principle in reality? If we look closely, the correspondence principle is like clothes of a naked king! Because in the Laplace or Fourier transform area, viscoelastic constitutive equations are equivalent to their corresponding elastic ones. It is often similar to Hooke's Law: $\sigma = \varepsilon E$. Also, for example, according to this principal, the viscoelastic constitutive equations can be found by using the operators method in form of the next equation, as

$$P(t)\sigma = Q(t)\varepsilon \tag{124}$$

where *P(t),* and *Q(t)* are linear operators, which usually can be represented as the sums of linear differential functions: $P(t) = \sum_0^n \frac{d^n}{dt^n} p_n$ and$Q(t) = \sum_0^n \frac{d^n}{dt^n} q_n$. But, for simple models *n = 0*, and it follows that $\frac{Q(t)}{P(t)} = C_{\sigma\varepsilon}$, and therefore

$$\sigma = C_{\sigma\varepsilon}\varepsilon \tag{125}$$

where $C_{\sigma\varepsilon}$ is the constant value! Thus, the correspondence principle also is applicable only for a small static or quasi-static deformation! Also, the main inconvenience of both the operator method and the Laplace transforms is the fact that for its use we should have an exact dependence of the elastic solution with including elastic constants.

Also, according to above-considered problems and solutions, we can allocate several main types of frictional contact as follows:

Purely elastic contact (which, in nature, practically does not exist): When only the elastic forces act between contacting surfaces. This does not exist in nature, but in some cases, if the mechanical losses are very small, we can consider the process of deformation as pure elastic!

Viscoelastic contact: When, between the contact surfaces, the dissipative forces of viscosity (also referred to as the forces of internal friction) also begin to act, but where there is no plastic deformation during the time of unloading! This is very often seen in nature and in solids and in liquids!

Elastic-plastic contact: When the forces of viscosity are considerable and the contacting surfaces pass into a plastic state!

Adhesive contact: When a significant adhesive force, which cannot be neglected, acts between the contact surfaces.

It is also important here to stress that all known models, such as Maxwell model, Standard Model and Zener model, and others, which build by joint an elastic element with the Maxwell element or the Kelvin element, are no match to viscoelastic behaviour, because in the moment of unloading, in the complete restitution of the main spring, the others elements are still in a deformed state! This is matched completely only by elastic-plastic behaviour!

In conclusion, it should be noted that the theory of Lee and Radok [8,9] is very problematic to use for viscoelastic materials at high deformation rates and especially at impact dynamic loads. The correspondence principle and Hooke's law are applicable only to static deformations and cannot be used for the dissipative force of viscosity and dissipative stresses, since for the occurrence of dissipative force there must be movement with some speed. There is a problem whereby, if the initial velocity is zero, the deformation will be delayed, as when the sample is stretched, and we observe viscoelastic hysteresis, but when impact occurs between bodies, the initial deformation velocity is not zero!!!

Yes, it can be assumed, for example, that when a sample with a flat section is stretched in an infinitely short period of time, the stress will be equal to the ratio between of the instantaneous force and the cross-sectional area. But, it is an impact too! Thus, in fact, we observe a creep and a stress relaxation like the result of an impact after sudden, very fast or instantaneous loading! Of course, after such a catastrophic impact, all the physical properties and parameters of matter will be very different from the equilibrium state. Therefore, the natural problem has arisen: How do I apply the theory of linear viscoelasticity in the event of an impact between curved surfaces? This is only possible if we consider infinitely small deformations in an infinitely small period of time, which was done using the "Method of specific forces" and the “Method of differential specific forces”, which are described in detail in articles [1,2] and in the book “Contact dynamics” [3].

Also, the term “Quasi-static” is so obscure for application, but it is often use by some scholars. The question arises: What is the difference between a quasi-static and a quasi-dynamic behaviour of materials!? Where is the border between them?

Now, let us imagine a high-rise building with no windows, just one entrance at the bottom! If you enter this building, you can see, in the low light, the carcass of an empty tower in which the old lower part has already become rusty, but the upper part is still being built. This carcass is for only one building and it still has not yet been fully completed! Installers continue to complete it and they have already reached the ceiling in which there is one single dark hatch through which you can go out onto the top of the roof! All the installers are at different levels, but everyone is very well equipped: they are all dressed in very good construction attire and all have safety harnesses that are secured to structural elements for safety! Installers can leave the building only back through the entrance or through an unknown dark hatch in the ceiling! But the problem is that in order to get out of the building, you need to take off the safety harness! Let me tell you a secret, if you go upstairs through the hatch, you can get to a beautiful city of physics and condensed matter physics in which there are a lot of large and beautiful science buildings with many doors and windows! Who do you think founded this carcass of an empty tower and who keeps building it!? Yes, your supposition is right, the base of this tower carcass which is still being built is the Radok & Lee’s theory of a quasi-static viscoelastic contact indentation.

The author hopes that the fundamental approach to existing problems set forth here and the solutions proposed here will be useful to all those who seek to use the physical essence of phenomena in their scientific research, and to use mathematics as a tool! Good luck!

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