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On The Symmetry Group Associated To Projection Operators Over A Probabilistic Space

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Abstract:
We Propose A Group Representation For The Operations That Selects A Sub-Space Of A Set Conformed By Mutually Dependent Probabilities Of Some Kind Of Events, We Call This Operations Projection Operators For The Analogy With The Projection Of A Vector In Some Particular Direction That We Can Define. We Started With The Minor Possible Level Of Dimension And Then We Expand For Higher Dimensions. So, We Show An Isomorphism Between The Operators And A Matrix Representation. The Goal Is To Suggest A Generalization To Other Kind Of Projection Operators Like Some Integral Ones And Show The Possible Applications To The Study Of The Properties Of The Fredholm Integral Equation In Electromagnetics.
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I. Introduction

From Statistical Mechanics to Quantum Mechanics the concept of a space of probabilities has been taken as the association of the points in either a phase space or an ordinary one, and a function that represents the probabilistic density of the existence of a piece from a set constituted of an ensemble of systems which are compatible versions of a particular one in the sense of a behavior under specific limits of energy, momentum, angular momentum, etc. But we can take a different point of view that is we can think in a space constituted by the probabilities that a system is in a specific state, that is, each point is the probability that the system be in a specific state. For an example, we can think in a system that can be found in only two different states, say state 1 and state 2, and that the respective probabilities for each state are p and q, so p + q must sum 1. The space is the set of the two p and q probabilities. Suppose that p= o.3 and q=0.7. Then the projection of the space in the event p can be found by taking the complete space 1 and subtracting q=0.7, that is 0.3. Also the corresponding projection of the space in the event q can be found subtracting p=0.3 from the entire space 1 that is q=0.7. But what about the projection over the entire space $p+q$ over himself? Let us define the operators:

$$\Omega_p \equiv -q \text{ And} \tag{1}$$

And $$\Omega_q \equiv -p\,, \tag{2}$$

And because $\mathrm{p}+\mathrm{q}=1$ then

$$\Omega_p = p-1. \tag{3}$$

If we try to operate

$$(p+q)1 \tag{4}$$

We must build

$$\Omega_p + \Omega_q = (p-1)+(-p) = -1 \tag{5}$$

Which is obviously incorrect. Now, we can try in other way:

$$(\Omega_p+\Omega_q)1 = \Omega_p 1 + \Omega_q 1 = (1-q)+(1-p)$$
$$=[p+q-(1-p)]+[p+q-p]=1 \tag{6}$$

That is: take 1 (the entire space) and subtract q so we have p; then we sum to this the result of now subtracting p to the entire space obtaining finally 1. So we must adjust the procedure to finally obtain the desired result 1. Before we propose even in a major dimension the convenient procedure we must note that the preceding two represents the fact that if we do not chose the appropriate rule then the conventional property of the complex and real numbers, vectors and matrices that is the associativity does not works for this kind of operators that we will call simply projection operators. Also we underline the possible application of the present work to the study of the Fredholm integral equations for which we have defined some integral projection operators in previous works [1].

II. The correct rule for adding projection operators

In this section we begin with a generalization of the rule for adding operators when we need to sum a higher number of them but first we will associate operators for the new probabilities and we denote the entire space as ξ that is:

$$\xi = \{p_i\} \qquad (7)$$

Where p_i is the probability to find the system in the i -th state and at the same time is a point of the set ξ and frequently we replace ξ for the complete probability to have the system in any state that is for the number 1:

$$\xi = 1 \qquad (8)$$

Let us denote:

$$\Omega_{p_i} \qquad (9)$$

For the corresponding projector operator over the state (or probability) p_i then, the sum of m projectors when the dimension of the space is n must be defined like:

$$(\sum_{i=1}^{m} \Omega_{p_i})\xi \equiv \xi - \sum_{j\neq i}^{n} p_j \qquad (10)$$

Because by the moment it is not necessary to show the case $\mathrm{m} \neq \mathrm{n}$ we take for example the case $\mathrm{m} = \mathrm{n} = 3$:

$$(\sum_{i=1}^{3} \Omega_{p_i})\xi = \xi - \sum_{j\neq i}^{3} p_j = \xi - 0 = \xi \qquad (11)$$

For m=2, n=3 that is for the sum over only two projectors is:

$$(\sum_{i=1}^{2} \Omega_{p_i})\xi = \xi - \sum_{j\neq i}^{3} p_j = p_1 + p_2 + p_3 - p_3 = p_1 + p_2 \qquad (12)$$

It is necessary to eliminate a restriction that appears in definition (10). If we try to sum:

$$p_3 + p_1 \qquad (13)$$

That is we want to sum from $\mathrm{i} = 3$ to $\mathrm{i} = 1$ or symbolically:

$$\sum_{\substack{i=3 \\ i\neq 2}}^{1} \Omega_{p_i} \equiv p_3 + p_1 \qquad (14)$$

This can be also written as a special case (with m=1, s=3) of:

$$\sum_{\substack{i=s \\ m<s}}^{m} \Omega_{p_i} \equiv \sum_{[i=1]}^{[m]} \Omega_{[p_i]} \qquad (15)$$

Where the parenthesis $[]$ signifies that we change the conventional order $i = 1, 2, ..., m, ..., n$ by a permutation $i = [1],[2],...,[m],...,[n]$ which in the case m=2, n=3 is: $i = [1],[2],[3]$ goes to $i = 3,1,2$. So in the general case equation (10) can be rewritten as:

$$(\sum_{[i=1]}^{[m]} \Omega_{[p_i]})\xi \equiv \xi - \sum_{[j\neq i]}^{[n]} p_{[j]} \qquad (16)$$

III. Matrix representation for the projector operators

The preceding rule expressed in equation (16) is useful for the purpose of give an understanding for the projection operators but we can go farther and find matrix representations that may also give an augmented vision of the possible implications to more complex systems than those former taken into account. In the process we will find additional properties like group properties and

a dual space for this representations that must be interpreted but indeed have not known existence at this moment.
Let us associate to the space of probabilities for the case n=2 a vector notation:

$$\xi = \begin{pmatrix} p \\ q \end{pmatrix} \quad (17)$$

Now, we define de four following projection operators:

$$\Omega_p = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (18)$$

$$\Omega_q = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \quad (19)$$

$$\Omega_\phi = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (20)$$

$$\Omega_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(Identity)} \quad (21)$$

The multiplication table for the projectors (18-21) is:

	Ω_I	Ω_p	Ω_q	Ω_ϕ
Ω_I	Ω_I	Ω_p	Ω_q	Ω_ϕ
Ω_p	Ω_p	$-i\Omega_\phi$	$i\Omega_I$	Ω_q
Ω_q	Ω_q	$i\Omega_I$	$i\Omega_\phi$	$-\Omega_p$
Ω_ϕ	Ω_ϕ	Ω_q	$-\Omega_p$	$-\Omega_I$

(22)

We can see that also the following rule is accomplished

$$\left(\Omega_{\mathrm{i}}\right)^4 = \Omega_{\mathrm{I}} \text{ With } i = I, p, q, \phi \quad (23)$$

So the set of four matrices:

$$\chi = \left\{\Omega_p, \Omega_q, \Omega_I, \Omega_\phi\right\} \quad (24)$$

Form an abelian cyclic group of fourth degree [2].
But we can verify that if we make the following associations:

$$p \Rightarrow \begin{pmatrix} p \\ iq \end{pmatrix} \equiv \left(1+i)\right) p \quad (25)$$

$$q \Rightarrow \begin{pmatrix} ip \\ q \end{pmatrix} \equiv \left(1+i\right) q \quad (26)$$

Then, $p, q,$ and ξ also they stick to the same rule (16):

$$(\Omega_p + \Omega_q)\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} - (0) = \begin{pmatrix} p \\ q \end{pmatrix} = \xi \quad (27)$$

$$\Omega_p \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ iq \end{pmatrix} = \left(1+i\right) p \quad (28)$$

$$\Omega_q \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ip \\ q \end{pmatrix} = \left(1+i\right) q \quad (29)$$

Also, we can verify that we can express the complete space ξ as the following sum:

$$p + q = \xi \quad (30)$$

We can see that other possibilities exist:

$$\Omega_{\phi}\begin{pmatrix} p \\ q \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = i\begin{pmatrix} p \\ -q \end{pmatrix} \equiv \phi \quad (31)$$

And for example:

$$\Omega_{\phi}\phi = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i\begin{pmatrix} p \\ -q \end{pmatrix} = -\begin{pmatrix} p \\ q \end{pmatrix} = -\xi \quad (32)$$

From equations (30) and (31) we can interpret ϕ as a dual space of probabilities and Ω_{ϕ} as the projection operator over that dual space.

In the next section, we will apply the recent developments to the problem of electromagnetic broadcasting.

IV. Projectors for the Generalized Inhomogeneous Fredholm

On electromagnetics we have developed an integral equation for the specific goal to study the broadcasting phenomena thanks to the procedure of successive approximations [1]. The former equation has been called the Generalized Inhomogeneous Fredholm Equation (GIFE). The fact is that simultaneously, the GIFE describes both a conventional travelling wave but also a resonance [3,4,5] which comes from the braking of the confinement of the evanescent waves. So we can think in the propagation media as a system which can have two possible states that is the state of the normal waves and that of the resonances. But we have defined a pair of projection operators that allow us to select between these two types of states. Then it seems that we can establish an isomorphism between the real integral projection operators and a matrix representation and also an appropriate representation for each broadcasting state; to this end, we recall the GIFE [1,6]:

$$s^{m}(\mathbf{r},\omega) = s^{m(\circ)}(\mathbf{r},\omega) + \Theta^{m}(\mathbf{r},\omega)$$

$$+\nu(\omega)\int_{0}^{\infty} \mathbf{K}_{n}^{m(\circ)}(\omega;\mathbf{r},\mathbf{r}')s^{n}(\mathbf{r}',\omega)dr' \quad (33)$$

In equation (33) the inhomogeneous term is called the generalized source (GS) that is:

$$\text{generalized source} \equiv s^{m(\circ)}(\mathbf{r},\omega) + \Theta^{m}(\mathbf{r},\omega) \quad (34)$$

This GS is indeed a blend of integral operators that has the role to allow access alternatively to either the resonances or the conventional travelling waves. Because of the properties of the GS, we can define the denominated Zap projection operators [1] (From zap or delete):

$$\square_{P}(\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega)) = \lim_{\eta\to 1} \mathrm{Z}(\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega)) \quad (35)$$

And

$$\square_{Q}(\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega))u^{m}(\mathbf{r};\omega) \equiv \lim_{\eta\to 1} \mathrm{Z}^{C}(\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega)) \quad (36)$$

(The exponent C means the complementary space)

In both equations (35) and (36) we have defined $\mathrm{Z}\left[\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega)\right]$ by the equation:

$$Zu^{m}(\mathbf{r};\omega) \equiv \mathrm{Z}\left[\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega)\right]u^{m}(\mathbf{r};\omega) =$$

$$+\Delta(\eta)\left[\eta - u^{m(\circ)}(\mathbf{r};\omega)\right]$$

$$+\eta\int_{0}^{\infty} \mathbf{K}_{n}^{m(\circ)}(\omega;\mathbf{r},\mathbf{r}')u^{n}(\mathbf{r}';\omega)dr' \quad (37)$$

We want underline that the argument of Z include the source term $u^{m(\circ)}(\mathbf{r};\omega)$ indicating that in each calculation an appropriate source must be considered.

The effect of application of the operators (35) and (36) is:

$$\square_{P}\left[\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega)\right]u^{m}(\mathbf{r};\omega) = y_{e}^{m}(\mathbf{r};\omega) \quad (38)$$

And

$$\square_{Q}(\mathbf{r};\omega;u^{m(\circ)}(\mathbf{r};\omega))y_{e}^{m}(\mathbf{r};\omega) = u^{m}(\mathbf{r};\omega) \quad (39)$$

The functions $y_e^m(\mathbf{r};\omega)$ and $u^m(\mathbf{r};\omega)$ are the solutions for the GIFE equation (equation (33)) and are respectively the resonant and the conventional travelling waves. By using the same technique as in section 3, we find matrix and vector representations for the projectors and for the solutions, so we can write in abbreviated form the equations (38) and (39):

$$\square_p \xi^m = (1+i)\, \overset{m}{p_e} \qquad (40)$$

$$\square_q \xi^m = (1+i)\, \overset{m}{q} \qquad (41)$$

Where:

$$\overset{m}{p_e} = \left(\frac{1}{1+i}\right)\begin{pmatrix} [p]\, y_e^m(\mathbf{r};\omega) \\ i[q]\, u^m(\mathbf{r};\omega) \end{pmatrix} \qquad (42)$$

$$\overset{m}{q} = \left(\frac{1}{1+i}\right)\begin{pmatrix} i[p]\, y_e^m(\mathbf{r};\omega) \\ [q]\, u^m(\mathbf{r};\omega) \end{pmatrix} \qquad (43)$$

$$\xi^m = \begin{pmatrix} [p]\, y_e^m(\mathbf{r};\omega) \\ [q]\, u^m(\mathbf{r};\omega) \end{pmatrix} \qquad (44)$$

$$\square_p \equiv \square_P \Omega_p \qquad (45)$$

$$\square_q \equiv \square_Q \Omega_q \qquad (46)$$

The operators Ω_p and Ω_q are the same as in equations (18) and (19) so we can complete the set of four projectors with those in equations (20) and (21) adding the two definitions:

$$\square_\phi \equiv \Omega_\phi \qquad (47)$$

$$\square_I \equiv \Omega_I \qquad (48)$$

Which gives:

$$\square_\phi \xi^m = i\begin{pmatrix} [p]\, y_e^m(\mathbf{r};\omega) \\ -[q]\, u^m(\mathbf{r};\omega) \end{pmatrix} \equiv \overset{m}{\phi} \qquad (49)$$

$$\square_I \xi^m = \xi^m \qquad (50)$$

In equation (49) we define the dual space $\overset{m}{\phi}$

In addition, it is possible verify that

$$\square_\phi \overset{m}{\phi} = -\xi^m \qquad (51)$$

Of course it is true that

$$p + q = 1 \qquad (52)$$

And we can find that $\overset{m}{p_e} + \overset{m}{q} = \xi^m$

So we can suppose that a hidden symmetry exist expressed in table (22) and a dual space $\overset{m}{\phi}$.

But now, because of equations (38) and (39) we can realize a very important simplification that is, we can redefine:

$$\square_P \Omega_p \to \Omega_p \equiv \square_p \qquad (53)$$

$$\square_Q \Omega_q \to \Omega_q \equiv \square_q \qquad (54)$$

With these definitions, we can see that the set of four projection operators:

$$\zeta = \left\{ \square_{p}, \square_{q}, \square_{\phi}, \square_{I} \right\} \qquad (55)$$

Form an abelian cyclic group of fourth degree [2]:

$$\left(\square_{i} \right)^{4} = \square_{I} \text{ With } i = p, q, \phi, I \qquad (56)$$

V. Conclusions

We have shown that the projection operators defined over a space of probabilities have a hidden symmetry that can be useful in some physical problems given us a more compact description as in the example for the so called Zap Projection Operators (Equations (40) and (41)) in electromagnetic broadcasting in which the matrix projection operators associated satisfy the multiplication table (22) developing an abelian cyclic group of fourth degree. Furthermore, we have proven that the matrix representations of the called Zap projection operators (Equation (55)) belong to the above referred cyclic group satisfying equation (56).

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