IOSR Journal Of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 17, Issue 1 Ser. 1 (Jan. – Feb. 2025), PP 25-29
www.Iosrjournals.Org

Application Of The Dynamical System Theory For Counting Black Hole Entropy Of Microstates

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Abstract:
The superstring theory is presented as one of the most advanced theories in physics that attempts to unify all four fundamental forces of nature into one single theory. Mathematical models in super-string theory due to their geometric nature describe spacetime as a multi-dimensional space in which strings can move. This allows to explain many properties of space-time, such as black-hole solution. Using the black-hole solution of general relativity, Kerr solutions, we received the Bekenstein-Hawking entropy, which can be generalized to spacetimes with dimension D. Analysis of the mechanism by which the black hole has entropy makes it possible to carry out calculations of such topological invariants as Lifschitz number and Euler characteristic of a black hole, represented as a multidimensional cubic space.
Keyword: *Black hole, Hawking radiation, theory of dynamical systems, the Lifschitz number and Euler characteristic of a black hole, multidimensional cubic space, topological entropy.*

Date of Submission: 10-01-2025 Date of Acceptance: 20-01-2025

I. Introduction

Superstring theory is one of the most advanced theories in physics that attempts to unify all four fundamental forces of nature into one single theory. It is based on the idea that all elementary particles and forces in nature can be explained as vibrations of ultramicroscopic strings. Mathematical models in superstring theory have their own unique properties and applications. They make it possible to describe various physical phenomena and processes, such as gravity, electromagnetism, strong and weak nuclear interactions.

One of the main properties of mathematical models in superstring theory is their geometric nature. They describe spacetime as a multidimensional space such as D-branes, BPS states, black holes in which strings can move. Hawking[1], Maldasina[2], Susskind[3] and other scientists studied the properties of these objects within the framework of the general theory of relativity and string theory. However, since the physical picture of processes and phenomena at high energies is quite complex and not completely clear, the mathematical apparatus for describing objects such as black holes, singularities of the early stages of the Universe and processes with elementary particles at high energies is not sufficiently developed. In this aspect, the possibilities of the theory of dynamical systems are promising, and the apparatus of topological invariants makes the theory computable. So, the use of dynamical systems is a new and effective apparatus for calculation of topological invariants that shed light on the properties of these exotic objects associated with the study of the early stages of the evolution of the Universe.

Our goal is to use the modern mathematical apparatus of the theory of dynamical systems to calculate the topological invariants of physical systems such as black holes. This allows us to explain many properties of space-time, such as its topology as well as to calculate invariants associated with physical observable quantities, such as the number of generations at high energies.

II. Geometric Nature Of Superstring Theory

In accordance with modern concepts of high-energy physics, the Universe is a combination of strings, the smallest parts of the Universe or specific vibrational pattern at Planck scale whose energy is of the order of 10^{28} eV, length is of the order of 10^{-35}m, time is of the order of 10^{-44}. In accordance with modern concepts of high-energy physics, the Universe is a combination of strings, the smallest particles of the Universe, which represent an ever-deeper penetration into matter from molecules, atoms, nuclei, quarks to strings.

The use of mathematical models in superstring theory also makes it possible to study the processes of birth and decay of elementary particles, as well as to explain the properties of black holes and other exotic objects. For example, the most well-known black-hole solution of general relativity is the Schwarzchild solution which gives in spherical polar coordinates the following metrics representing the line element outside of a body of mass M:

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2MG}{r}\right)}dr^2 + r^2 d\Omega_2^2$$

This model allows us to describe the properties of a black hole, such as its radius, r, mass, M. The Schwarzschild metric is the spherically symmetric exact solution of Einstein's equations without a cosmological constant in empty space, due to Birkhoff's theorem. This result can be generalized to black holes with electric charge Q by solving the Einstein-Maxwell equations. They are derived from the action

$$S_{EM} = \frac{1}{16\pi G_N}\int^{d^4} x(-g)^{\frac{1}{2}}\left[R - G_N F_{\mu\nu}F^{\mu\nu}\right]$$

The only static solution is the Reissner-Nordstrom (RN) solution with line element

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega_2^2.$$

The stationary solutions of the Einstein equations are the (axially-symmetric) Kerr solutions, classified by two parameters, the mass M and the angular momentum J. Generalizing to solutions of the Einstein-Maxwell equations leads to the three-parameter Kerr-Newman metrics:

$$ds^2 = \frac{\Delta - a^2 \sin^2 \theta}{\Sigma}dt^2 + sa\sin^2 \theta\frac{r^2 + a^2 - \Delta}{\Sigma}dtd\phi -$$
$$-\frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2 \theta}{\Sigma}\sin^2 \theta d\phi^2 - \frac{\Sigma}{\Delta}d^2 - \Sigma d\theta^2,$$

where

$$\Sigma \equiv r^2 + a^2\cos^2 \theta,\ \Delta \equiv r^2 - 2Mr + a^2 + Q^2.$$

The parameters M, a, Q are related to the total angular momentum J by

$$a = \frac{J}{M}.$$

If such a black hole is perturbed and settles down to another stationary black hole with parameters M + dM, J + dJ and Q + dQ, then

$$dM = \frac{k}{8\pi}dA_H + \Omega_H dJ + \Phi_H dQ. \quad (1)$$

The first law thermodynamical formula is the following

$$dU = TdS + pdV + \mu dN. \quad (2)$$

Comparing expressions (1) and (2) and taking into account that quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with a temperature T_{bh} given by

$$T_{bh} = \frac{k}{2\pi}$$

we can get Bekenstein-Hawking entropy

$$S_{bh} = \frac{1}{4}A_H$$

Using Feynman's path integral formulation of quantum field theory and Smarr's formula we received the Bekenstein-Hawking entropy

$$S = \frac{c^3 A_H}{4G_N \hbar},$$

which can be generalized to spacetimes with dimension D

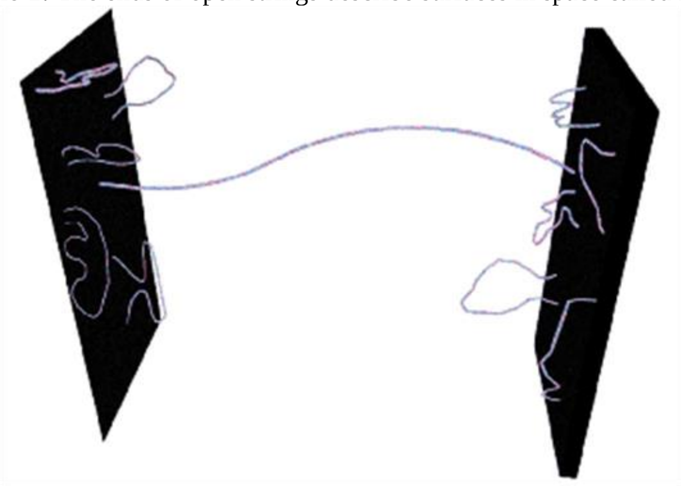
$$S = \frac{c^3 A_D}{4G_D \hbar},$$

where A_D is the (D - 2)-dimensional 'area' of the event horizon and G_D is the D-dimensional Newton constant.

III. The Mechanism By Which A Black Hole Has Entropy

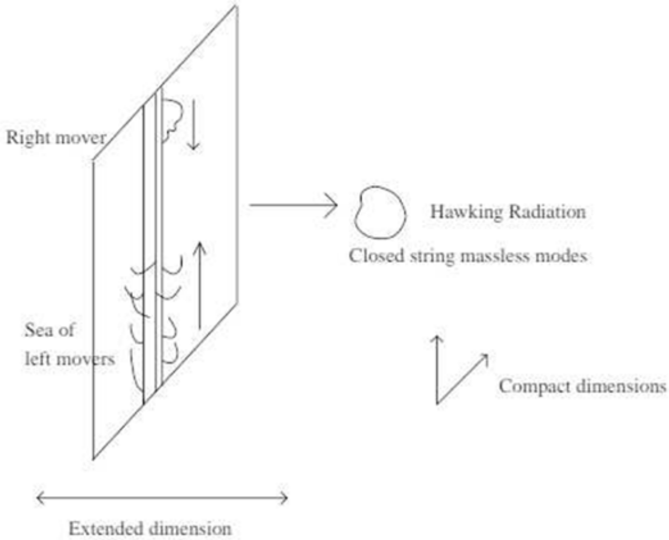
According to confirmed experimental data, a black hole is a dynamic system that absorbs and emits. So, calculating its area even in multidimensional space requires a certain correction and precision approach, taking into account the dynamics of string objects inside it. Principles of quantum mechanics and thermodynamics demonstrate that there are many hidden degrees of freedom that give the black hole its entropy. Susskind's work states that the classical Bekenstein entropy of a black hole is argued to arise from configurations of strings with ends which are frozen on the horizon[4]. It was proposed that the area of the stretched horizon rather than the event horizon should be identified with the entropy. So, for a class of BPS extremal holes it is supposed that all black holes are single string states as the level density of strings reproduces the Bekenstein entropy[5]. In physical systems the entropy has a statistical interpretation in terms of counting microscopic configurations and this counting requires an understanding of the quantum degrees of freedom. For black holes this is a puzzle connected with its degrees of freedom, which was resolved thanks to string solitons - "D-branes", Figure 1.

Figure no 1: The ends of open strings describe surfaces in space called D-brane.



Hawking radiation is described by open strings colliding and forming a closed string. Configuration of D-branes in compact dimensions with open D-strings between D-fivebranes is presented in Figure 2.

Figure no 2: D-brane picture of the Hawking radiation emission process.



The string entropy is realized through the multidimensional lattice of points with the strings inside it, which can move in any of 2d directions. So, according to Susskind, the entropy can be calculated by the following formula,[4]

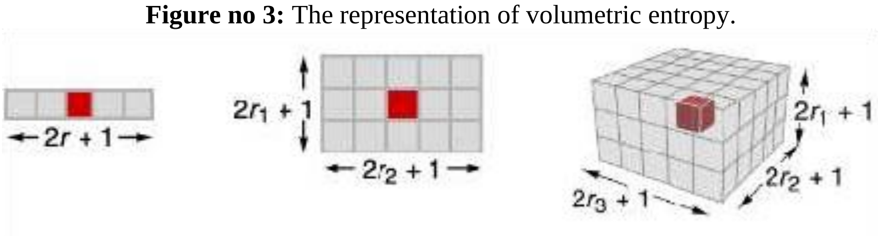
$$S_s = \ln(2d)^n = n \cdot \ln 2\,d$$

The calculation of entropy assumes the calculation of all degrees of freedom of the coordinate system under consideration,

$$S = \log(2r_1 + 1)\,(2r_2 + 1)(2r_3 + 1),$$

as presented in Figure 3.

Figure no 3: The representation of volumetric entropy.



A schematic representation of volumetric entropy through multi-colored cubes (analogs of different string orientations) in multidimensional space is impossible, but in three dimensions the entropy calculations are implemented in the Mathematica package [6], which is shown schematically in Figure 4.

Figure no 4: The schematic representation of entropy filter with smaller neighbors.

The above calculation presented quite generic point in string theory: the true degrees of freedom at high energies may be different than one expects from perturbative considerations. For critical string theory we don't know about these degrees of freedom. Therefore, it is advisable to consider the application of dynamical systems theory that describes a state space and an evolution rule that specifies how the state changes over time. The state space can be a manifold, and the evolution rule can be differential equations, so dynamical system is described as a particle whose state varies over time and obeys differential equations. So, we can consider differentiable mappings connected with properties of different manifolds in terms of issues a) the description of a mapping in a neighborhood of a critical point; b) the description of the structure of the set of critical points. As there are no answers to a) and b) for obtaining explicit results it is sufficient to know the answers for only a large set of mappings for further quantitative information about the string system. Therefore, to display the complex dynamic process of string geometry inside a black hole, it is necessary to consider the evolution rule as a map of continuous state space of strings. As we have considered multidimensional lattice of points with the strings inside it, then we are dealing with mapping of spaces

$$R^{2d} \to R^{2d}, \quad d = 1, \dots, n.$$

As d is sufficiently large, this mapping can be presented by $f: S^{2d} \to S^{2d}$. There is the known result, that in the special case of a mapping $f: S^n \to S^n$ from the n-sphere to itself, the degree is simply the winding number of the mapping[7], and therefore,

$$\deg f = 2d.$$

In accordance with Misiurewicz-Przytycki Theorem for smooth compact orientable manifold

$$h_{top}(f) \geq \log|\deg(f)|.$$

Topological entropy of the system of n links of string length, $L = l_s n$, is the n sum,

$$h_{top} = n \cdot \log(2d).$$

Any continuous mapping of a sphere onto itself has isolated point, according to the fixed-point theorem proved in 1912 by Brouwer [8]. Isolated fixed points are characterized by index and according to Lifschitz Fixed-Point Formula for compact manifolds with continuous mappings $f: M \to M$

$$L(f) = \sum_{x \in F_{ix}(f)} \mathrm{ind}_f x.$$

For f homotopic to the identity

$$L(f) = \sum_{k=0}^{\dim M} (-1)^k \beta_{k.}$$

Using Euler-Poincare Formula

$$\sum_{k=0}^{\dim M} (-1)^k \beta_k = \chi(M)$$

The Euler characteristics of M for maps homotopic to identity with only isolated fixed points in accordance with the Hopf-Poincare index theorem

$$\sum \mathrm{ind}_f x = \chi(M).$$

Using the Proposition 8.6.8.[9] we can conclude, that the Lifschitz number

$$L(f^n) = 1 + (-1)^m \deg f^n$$

of one link is equal to

$$L(f^n) = 1 + (2d)^n.$$

Therefore the Euler characteristics of S^n mappings is equal to

$$1 + (2d)^n = \chi(M).$$

Thus, using the theory of dynamical systems, we calculated the entropy, Lifschitz number and Euler characteristic of a black hole, represented as a multidimensional cubic space. This information is important because of the connection of Euler characteristic with hodge numbers

$$\chi = \sum_{p,q=n} (-1)^{p+q} h^{p,q} = 2(h^{1,1} - h^{2,1}).$$

and the number of vector and scalar multiplets in vacuum, which are associated with phase transitions at high energies[10].

IV. Conclusions

We presented an interpretation of a black hole in terms of a collection of D-branes of different types, the Hawking radiation of which is carried out due to the radiation of closed strings. This interpretation of black holes allows us to imagine quantum degrees of freedom in terms of putting many open strings on the D-brane. The calculation of the string entropy is realized through the consideration of a multidimensional lattice of points with the strings inside it, which can move in any of 2d directions. Within the framework of the theory of dynamical systems, we carried out calculations of topological entropy in accordance with the fixed-point theorem. Considering the Hopf-Poincare theorem, we calculated the Lifschitz number and Euler characteristic of a black hole, represented as a multidimensional cubic space. These characteristics are interpreted in terms of elementary particles, since the Euler characteristic is related to the number of generations of quarks and leptons.

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