IOSR Journal Of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 17, Issue 2 Ser. 3 (Mar. – Apr. 2025), PP 15-18
www.Iosrjournals.Org

A Generalized Class Of Branched Hamiltonians

Dibyendu Ghosh
Department Of Mathematics, Barrackpore R. S. N College
Barrackpore, Kolkata-700120, India

Abstract
Hamiltonian that are multi-valued structure of momenta present are topical interest since they correspond to the Lagrangians containing higher degree time derivatives. The governing Hamiltonians namely, the branched Hamiltonians are explored, as recently advocated by Shapere and Wilczek.
Keywords: *Branched Hamiltonian, multi-valued, quantum.*

Date of Submission: 25-03-2025 Date of Acceptance: 05-04-2025

I. Introduction

Lagrangians are convex functions of velocity as described by conventional physical systems. Therefore, a unique single-valued Hamiltonian can be obtained by applying the Legendre transformation and this results in a well-defined general formalism of classical and quantum theories.

Their Lagrangians are, therefore, convex functions being quadratic in velocities with positive coefficient. But what is more interesting is the physical systems with non-convex Lagrangians. It is all because they are significant for studying spontaneous breaking of time translation symmetry [1, 2, 3] and are widely applied to theories of cosmology, optics, shallow water-wave studies, gravitation, and certain types of quantum mechanical models. Under conventional Legendre transformation, systems with a non-convex Lagrangian will result in a multi-valued Hamiltonian as a function of conjugate momentum. Therefore, the study of the proposed research is related to the multi-valued structure of the underlying Hamiltonians of the mechanical system.

As a part of the the methodology and organization of work elements in the present research project, we need to search for the initial attempts where multi-valued Hamiltonians have appeared that result from the enforcement of the Legendre transform on the non-standard Lagrangians in the sense that the velocity dependence is not convex. This would also make the topological aspects an interesting side issue.

In a series of papers [4,5,6] the problem of branched Hamiltonians was looked at those results from non-standard forms of Lagrangians (NSL). The latter are standard if they can be expressed as the difference between the kinetic energy and potential energy terms, whereas the NSL may not follow such a prescription.

The motivation came from an earlier work of Curtright and Zachos [4] who interpreted a classical system described by a pair of convex, smoothly tied functions of the velocity variable v. It was shown to possess a quantum counterpart that exhibits a double-valued Hamiltonians similar to a supersymmetric system in the momentum space.

II. A Generalized Class Of Branched Hamiltonians

Let us discuss the example of a branched system as illustrated in [4]. It is to be noted here that a typical classical model of the above-mentioned non-quadratic type may be sampled by Lagrangian

$$L = C(v-1)^{\frac{2k-1}{2k+1}} - V(x), \text{ where } C = \frac{2k+1}{2k-1}\left(\frac{1}{4}\right)^{\frac{2}{2k+1}} \tag{1}$$

Function $V(x)$ represents a convenient local interaction potential while the traditional kinetic-energy term is tentatively replaced by a fairly unusual function of "velocity" v.

This model was recently analyzed in Ref. [4] where the definition of the fractional powers of difference $v-1$ was adapted to the needs of possible phenomenology. In detail, the $(2k+1)$-st root was required real and positive or negative for $v>1$ or $v<1$, respectively. By doing this we are in fact taking the real parts of two different branches of the analytic $(2k+1)$-st root as a function of complex v. We do this solely to have a real, single-valued Lagrangian function for all real v.

Under the circumstances, there is every reason to draw on more than one branch of an analytic function of v provided that only one branch is encountered at any given real v, or at least that would seem to be true for classical dynamics. The following graph, will illustrate the consequences this choice for L has for the quantum dynamics is, especially for the case $k=1$.

If we expand the above Lagrangian for the case $k = 1$ by Taylor's series for v near zero, we then have $L \approx C\left(-1 + \frac{v}{3} + \frac{v^2}{9} + O(v^3)\right) - V(x)$. Of these terms, the first is naive, the second would give a boundary contribution to the action and therefore not effect the equations of motion, and the third is the usual v^2 kinetic structure:

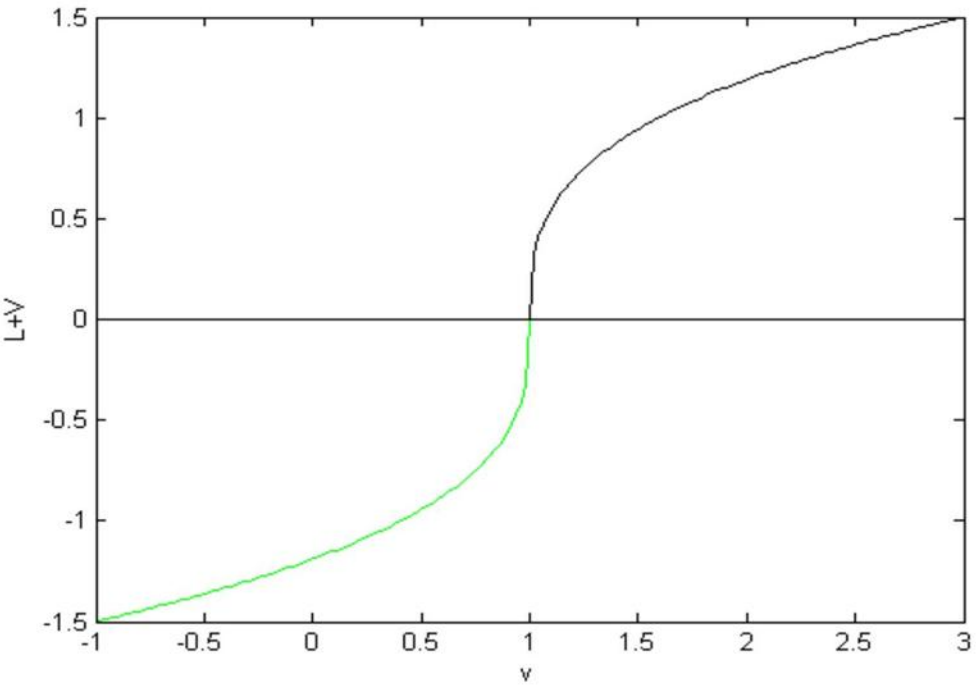

Figure 1: $L + V = C(v-1)^{\frac{1}{3}}$ for the case $k = 1$.

$$A = \int_{t_1}^{t_2} L dt$$
$$\approx C\left(t_2 - t_1 + \frac{1}{3}(x(t_2) - x(t_1)) + \frac{1}{9}\int_{t_1}^{t_2} v^2 dt + \int_{t_1}^{t_2} O(v^3) dt\right) - \int_{t_1}^{t_2} V(x) dt \quad (2)$$

So, this action would result in the usual Newtonian classical equations of motion for small v. On the other hand, for large velocities, the v dependence is more elaborate, leading (for finite, positive integer k) to a non-convex function of velocity, whose curvature $\partial^2 L/\partial v^2$ flips sign at just one point, namely, $v = 1$.

Thus, the function L may be thought of a single pair of convex functions judiciously pieced together. The non-convexity of L has the effect of making the Kinetic energy, and hence the Hamiltonian, a double-valued function of p.

In the case of the Lagrangian (1), the canonical momentum turns out to be

$$p = \left(\frac{1}{4}\right)^{\frac{2}{2k+1}} \frac{1}{(v-1)^{\frac{2}{2k+1}}}, \tag{3}$$

whose inversion immediately tells us that the velocity variable $v(p)$ is a double-valued function of p.

$$v_{\pm}(p) = 1 \mp \frac{1}{4}\left(\frac{1}{\sqrt{p}}\right)^{(2k+1)} \tag{4}$$

This has the implication that if we evaluate the Hamiltonian its branches $H_{\pm}$will appear

and hence the Hamiltonian, a double-valued function of p. For any positive integer k, we find two branches for H,

$$H_{\pm} = p \pm \frac{1}{4k-2}\left(\frac{1}{\sqrt{p}}\right)^{2k-1} + V(x) \tag{5}$$

For $k = 1$, the two kinetic energy branches have the shape shown in the figure below. Note that, classically, p must be non-negative for this model to avoid imaginary $v(p)$. That is to say, the slope $\frac{\partial L}{\partial v}$ is always positive.

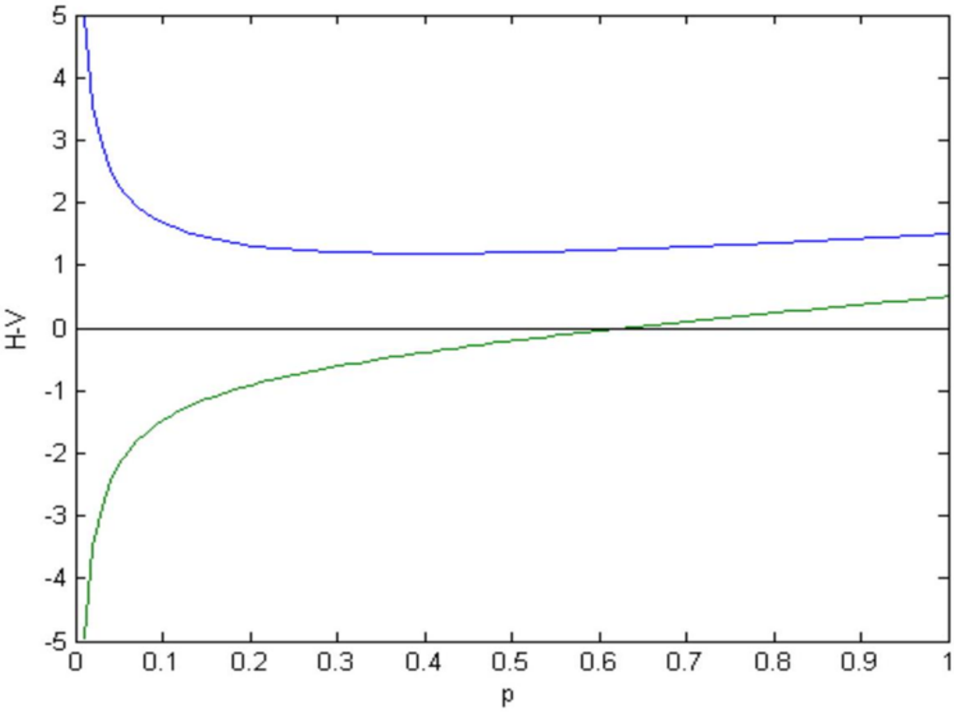

Figure 2: $H_{\pm} - V(x) = p \pm \frac{1}{2\sqrt{p}}$ in blue/green. There is a cusp at $p = \infty$.

We have also noted that $k = 1$ case speaks of the canonical supersymmetric structure [7] for the difference $H_{\pm} - V(x)$ namely, $p \pm \frac{1}{2\sqrt{p}}$, but in the momentum space if viewed as a quantum mechanical system. The spectral and boundary condition linkages of these Hamiltonian are not difficult to set up.

III. Some Classes Of Branched Hamiltonians

In practice, the price to pay for having the model quantized lies in the clarification of the role of singularities in the eigenfunction. In this sense the road to quantization is, expectedly, not free of difficulties. One can, however, feel encouraged by the work by Chithiika Ruby et al. [8] who showed that the energy spectrum and normalized solutions could still be obtained for a class of Hamiltonians that are nonsymmetric and non-Hermitian.

Needless to add that the non-Hermitian nature of quantum Hamiltonian may bring about a number of unpleasant consequences as, for example, the emergence of the exceptional points [9] or the breakdown of the adiabatic theorem [10, 11]. At the same time, the acceptance of the anomaly may prove innovative, e.g., by giving a new physical interpretation to wave packets [12] or to pseudospectra [13, 14].

In a way inspired by Eq. (1) let us consider a higher-power Lagrangian

$$L = C(v + f(x))^{\frac{2m+1}{2m-1}} - \delta, C = \left(\frac{1-2m}{1+2m}\right)(\delta)^{\frac{2}{1-2m}}, \delta > 0. \tag{6}$$

The main difference from (1) lies in our choice of a general function $f(x)$ in place of $f(x) = -1$ in our initial L. The other point is that the inverse exponent with respect to the model of Curtright and Zachos [4] is done for convenience of calculus. Further, we have omitted an explicit presence of the potential function assuming that the interaction re-appears in a more natural manner via a suitable choice of an auxiliary free parameter δ and of a nontrivial function $f(x)$. As long as our Lagrangian L is of a nonstandard type, we will not feel disturbed by the absence of the explicit potential $V(x)$.

Parameter C is non-negative for $0 \leq m < \frac{1}{2}$ and the canonical momentum is given by formula

$$p = \frac{\partial L}{\partial v} = -(\delta)^{\frac{2}{1-2m}}(v + f(x))^{\frac{2}{2m-1}} \tag{7}$$

which can easily be inverted to yield

$$v = -f(x) + \delta\left(\pm\sqrt{-p}\right)^{2m-1} \tag{8}$$

The Hamiltonian has the structure

$$H_{\pm}(x,p) = (-p)f(x) - \frac{2\delta}{2m+1}(\pm\sqrt{-p})^{2m+1} + \delta \tag{9}$$

For the present purposes it is worthwhile to inquire into the specific case with $m = 0$. At $m = 0$ we easily derive the double-valued

$$v = v_{\pm} = -f(x) \pm \frac{\delta}{\sqrt{-p}} \tag{10}$$

The Hamiltonian branches out to the components

$$H_{\pm}(x,p) = (-p)f(x) \mp 2\delta\sqrt{-p} + \delta \tag{11}$$

It is readily noticed that the real or the complex character of $H_{\pm}$depends on the sign of the momentum p. Once we specify

$$f(x) = \frac{\lambda}{2}x^2 + \frac{9\lambda^2}{2k^2}, \delta = \frac{9\lambda^2}{2k^2}, \lambda > 0 \tag{12}$$

then under a shift $p \to \frac{2k}{3\lambda}p - 1, H_{\pm}$move over to the corresponding forms

$$H_{\pm}(x,p) = \frac{9\lambda^2}{2k^2}\left[2 \mp 2\left(1 - \frac{2kp}{3\lambda}\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\lambda} - \frac{2kp}{3\lambda} - \frac{2k^3x^2p}{27\lambda^2}\right] \tag{13}$$

These represent a set of plausible Hamiltonians for the nonlinear Liénard system [15, 16,17]. In both $H_{\pm}$ the presence of a linear harmonic oscillator potential is revealed in the limit $k \to 0$. In the classical scenario one needs to restrict p to $-\infty < p \le \frac{3\omega^2}{2k}$ in order to address the physical properties of the system in the real space. However, the presence of a branch point singularity at $p = \frac{3\lambda}{2k}$ makes the study of $H_{\pm}$quite complicated. At the boundary $p = \frac{3\lambda}{2k}$, the two Hamiltonians $H_{\pm}$coincide.

IV. Summary

To summarize, we have looked at a model of nonlinear oscillator describing quasiharmonic oscillations and showed the Hamiltonian suited for it has a non-conventional double-valued structure due to the presence of a velocity-dependent potential.

References

[1] A. Shapere And F. Wilczek, Phys. Rev. Lett. 109, 200402 (2012).
[2] A. Shapere And F. Wilczek, Phys. Rev. Lett. 109, 160402 (2012).
[3] F. Wilczek, Phys. Rev. Lett. 109, 160401 (2012).
[4] T. L. Curtright And C. K. Zachos, J. Phys. A: Math. Theor. 47, 145201 (2014).
[5] B. Bagchi, S. Modak, P. K. Panigrahi, F. Ruzicka And M. Znojil, Mod. Phys. Lett. A 30, 1550213 (2015).
[6] B. Bagchi, S. M. Kamil, T. R. Tummuru, I. Semoradova And M. Znojil, J. Phys.: Conf. Series 839, 012011 (2017).
[7] E. Witten, Nucl. Phys. B 188, 513 (1981).
[8] V. Chithiika Ruby, M. Senthivelan And M. Lakshmanan, J. Phys. A: Math. Theor. 45, 382002 (2012).
[9] F. Bagarello And F. Gargano, Phys. Rev. A 89, 0321132 (2014).
[10] K. Hashimoto, K. Kanki, H. Hayakawa And T. Petrosky, Prog. Theor. Exp. Phys. 2015, 023a02 (2015).
[11] T. J. Milburn, J. Doppler, C. A. Holmes, S. Portolan, S. Rotter And P. Rabi, Phys. Rev. A 92, 052124 (2015).
[12] E. M. Grafe, H. J. Korsch, A. Rush And R. Schubert, J. Phys. A: Math. Theor. 48, 055301 (2015).
[13] R. Novak, Int. J. Theor. Phys. 54, 4142 (2015).
[14] F. Ruzicka, Int. J. Theor. Phys. 54, 4154 (2015).
[15] A. Liénard, Revue Générale De L'électricité 23, 901 (1928).
[16] A. Liénard, Revue Générale De L'électricité 23, 946 (1928).
[17] M. Lakshmanan And S. Rajasekar, (2013) Nonlinear Dynamics: Integrability Chaos And Patterns, Springer-Verlag, New York.