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Representations and Computation of the Nield- Kuznetsov Integral Function

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Abstract: *The Nield-Kuznetsov integral function is studied in this work and some of its additional properties and representations are discussed. In particular, its representations in terms of Bessel and Hankel functions are derived. Ascending series approximations are presented in a form more suitable for computations using Cauchy product. An initial value problem is formulated and solved using ascending series and asymptotic approximations.*

Keywords: *Airy's non-homogeneous equation, Nield-Kuznetsov function.*

I. Introduction

In their analysis of coupled, parallel flow, Nield and Kuznetsov [1] introduced the concept of a transition layer, which we define in the current context as a porous layer of variable permeability typically sandwished between a constant-permeability porous layer and a free-space channel. In order to obtain an exact solution to the flow problem, Nield and Kuznetsov [1] reduced Brinkman's equation which governs the flow through the transition layer to the well-known inhomogeneous Airy's equation. Their solution to Airy's equation resulted in the introduction of an integral function, referred to as the Nield-Kuznetsov function $Ni(x)$, [2], which continues to receive considerable attention in the literature due to its many mathematical properties (*cf.* [2,3,4]), its usefulness in fluid flow applications, [5], and its utility in the solution of initial and boundary value problems involving Airy's inhomogeneous equation with both constant and variable forcing function.

The literature reports on a large number of properties and representations of $Ni(x)$, its derivatives, and the many differential equations it satisfies, [2]. However, further studies of this function are still needed to discover its many applications and the different areas of the sciences in which it arises. In this work we discuss further properties of the Nield-Kuznetsov function and its use in the solution of initial and boundary value problems of the Airy's inhomogeneous equation. We offer various representations of this function, discuss its computational aspects, and develop its relationship to other integral functions.

II. Solution to Airy's Initial Value Problem

Consider the inhomogeneous Airy's ordinary differential equation (ODE), [6]:

$$\frac{d^2y}{dx^2} - xy = f(x) \tag{1}$$

subject to the initial conditions:

$$y(0) = \alpha \tag{2}$$

$$\frac{dy}{dx}(0) = \beta \tag{3}$$

Where α and β are known constants.

General solutions to ODE (1) are given as follows, for different forcing functions $f(x)$. When $f(x) = 0$, general solution to the homogeneous Airy's ODE is given

$$y = a_1 Ai(x) + a_2 Bi(x) \tag{4}$$

where a_1 and a_2 are arbitrary constants, and the functions $Ai(x)$ and $Bi(x)$ are two linearly independent functions known as Airy's homogeneous functions of the first and second kind, respectively, and are defined by the following integrals (*cf.* [7,8,9]):

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right)dt \tag{5}$$

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt + \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt. \tag{6}$$

The Wronskian of $Ai(x)$ and $Bi(x)$ is given by, [7]:

$$W(Ai(x), Bi(x)) = Ai(x)\frac{dBi(x)}{dx} - Bi(x)\frac{dAi(x)}{dx} = \frac{1}{\pi} \tag{7}$$

And the solution satisfying the initial conditions (2) and (3) takes the form

$$y = \pi\left\{\frac{3^{1/6}\alpha}{\Gamma(1/3)} - \frac{\beta}{3^{1/6}\Gamma(2/3)}\right\}Ai(x) + \pi\left\{\frac{\alpha}{3^{1/3}\Gamma(1/3)} + \frac{\beta}{3^{2/3}\Gamma(2/3)}\right\}Bi(x). \tag{8}$$

When $f(x) = -\frac{1}{\pi}$, general solution to ODE (1) is given by

$$y = b_1 Ai(x) + b_2 Bi(x) + Gi(x) \tag{9}$$

and for $f(x) = \frac{1}{\pi}$, general solution to ODE (1) is given by

$$y = c_1 Ai(x) + c_2 Bi(x) + Hi(x) \tag{10}$$

Where $b_1, b_2, c_1,$ and c_2 are arbitrary constants, and the functions $Gi(x)$ and $Hi(x)$ are known as Scorer's functions, [10,11], and are given by

$$Gi(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt \tag{11}$$

$$Hi(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt. \tag{12}$$

From (9) and (10) we obtain, respectively, the following solutions satisfying the initial conditions (2) and (3):

$$\begin{aligned} y = {} & \pi\left\{\frac{3^{1/6}\alpha}{\Gamma(1/3)} - \frac{\beta}{3^{1/6}\Gamma(2/3)}\right\}Ai(x) \\ & + \pi\left\{\frac{\alpha}{3^{1/3}\Gamma(1/3)} + \frac{\beta}{3^{2/3}\Gamma(2/3)} - \frac{1}{3\Gamma(1/3)\Gamma(2/3)}\right\}Bi(x) + Gi(x) \end{aligned} \tag{13}$$

$$\begin{aligned} y = {} & \pi\left\{\frac{3^{1/6}\alpha}{\Gamma(1/3)} - \frac{\beta}{3^{1/6}\Gamma(2/3)}\right\}Ai(x) \\ & + \pi\left\{\frac{\alpha}{3^{1/3}\Gamma(1/3)} + \frac{\beta}{3^{2/3}\Gamma(2/3)} - \frac{4}{3^{2/3}\Gamma(1/3)\Gamma(2/3)}\right\}Bi(x) + Hi(x) \end{aligned} \tag{14}$$

In applying the initial conditions (2) and (3) to obtain the values of arbitrary constants in (4), (9) and (10), we utilized the following values of the integral functions $Ai(x), Bi(x), Gi(x), Hi(x),$ and their first derivatives at x=0, reported in [7,8]. We list these values for convenience and reference in **Table 1**, below.

Integral Function at $x=0$	**Derivative of Integral Function at** $x=0$
$Ai(0)=\frac{\sqrt{3}}{3^{7/6}\Gamma(2/3)}=\sqrt{3}Gi(0)$	$\frac{dAi}{dx}(0)=\frac{-\sqrt{3}}{3^{5/6}\Gamma(1/3)}=-\sqrt{3}\frac{dGi}{dx}(0)$
$Bi(0)=\frac{3}{3^{7/6}\Gamma(2/3)}=3Gi(0)$	$\frac{dBi}{dx}(0)=\frac{3}{3^{5/6}\Gamma(1/3)}=3\frac{dGi}{dx}(0)$
$Gi(0)=\frac{1}{3^{7/6}\Gamma(2/3)}=\frac{Ai(0)}{\sqrt{3}}$	$\frac{dGi}{dx}(0)=\frac{1}{3^{5/6}\Gamma(1/3)}=-\frac{1}{\sqrt{3}}\frac{dAi}{dx}(0)$
$Hi(0)=\frac{2}{3^{7/6}\Gamma(2/3)}=\frac{2}{3}Bi(0)$	$\frac{dHi}{dx}(0)=\frac{2}{3^{5/6}\Gamma(1/3)}=\frac{2}{3}\frac{dBi}{dx}(0)$

Table1. Values of the integral functions and their derivatives at *x=0*.

We observe in the above solutions that the Scorer functions furnish the particular solution to inhomogeneous Airy's ODE for special values of $f(x)$. When $f(x)$ is a general constant or a variable function of *x*, the Scorer functions do not directly render the needed particular solution.

In order to provide for a more general approach to particular solutions to ODE (1), we apply the method of variation of parameters to obtain the following form of y when $f(x)=R$, where R is any constant:

$$y=m_1Ai(x)+m_2Bi(x)-\pi RNi(x) \tag{15}$$

where m_1 and m_2 are arbitrary constants and the function $Ni(x)$ is the Nield-Kuznetsov function, [1,2]. The function $Ni(x)$ is defined in terms of Airy's functions as, [1]:

$$Ni(x)=Ai(x)\int_0^x Bi(t)dt-Bi(x)\int_0^x Ai(t)dt \tag{16}$$

with a first derivative given by

$$\frac{dNi(x)}{dx}=\frac{dAi(x)}{dx}\int_0^x Bi(t)dt-\frac{dBi(x)}{dx}\int_0^x Ai(t)dt\,. \tag{17}$$

Equations (16) and (17) give the values $Ni(0)=\frac{dNi(0)}{dx}=0$, which we can use in determining the arbitrary constants in (15) when initial conditions (2) and (3) are used. Solution to the initial value problem thus takes the following form (in which we use $W(Ai(x),Bi(x))=-\frac{d^2Ni(x)}{dx^2}=-\frac{1}{\pi}$):

$$y=\pi\left\{\frac{3^{1/6}\alpha}{\Gamma(1/3)}-\frac{\beta}{3^{1/6}\Gamma(2/3)}\right\}Ai(x)+\pi\left\{\frac{\alpha}{3^{1/3}\Gamma(1/3)}+\frac{\beta}{3^{2/3}\Gamma(2/3)}\right\}Bi(x)-\pi RNi(x)\,. \tag{18}$$

We note at the outset that in solving the initial value problem and expressing the solution in terms of $Ni(x)$, values of the arbitrary constants are independent of the forcing function. To illustrate this point further, when $f(x)=\mp\frac{1}{\pi}$, solution to the initial value problem is simply given by equation (15) with $R=\mp\frac{1}{\pi}$. In other words, coefficients of Airy's functions do not change in this case.

III. Representations of Ni(x)

In order to study the behavior of $Ni(x)$ and its derivatives over a subset *[a,b]* of the real line, we need to evaluate this function at specified values of *x*. This is accomplished by utilizing their definitions in terms of Airy's functions. In what follows, we provide expressions for $Ni(x)$, its first derivative and its integral in terms of asymptotic and ascending series, and in terms of Bessel functions representation of $Ai(x)$ and $Bi(x)$.

3.1. Representations in terms of Scorer's and Airy's Functions

In this section we rely on the representations of *Ai(x), Bi(x), Gi(x)* and *Hi(x),* given by equations (5), (6), (11), and (12).

The following relationships have also been reported in the literature (*cf.* [7,8,9]), and are reproduced here for reference:

$$Gi(x) = Ai(x)\int_0^x Bi(t)dt + Bi(x)\int_x^\infty Ai(t)dt \tag{19}$$

$$Hi(x) = Bi(x)\int_{-\infty}^x Ai(t)dt - Ai(x)\int_{-\infty}^x Bi(t)dt \tag{20}$$

$$Gi(x) + Hi(x) = Bi(x) \tag{21}$$

$$\int_0^x Ai(t)dt = \frac{1}{3} + \pi\left\{Gi(x)\frac{dAi(x)}{dx} - Ai(x)\frac{dGi(x)}{dx}\right\} \tag{22}$$

$$\int_0^x Bi(t)dt = \pi\left\{Gi(x)\frac{dBi(x)}{dx} - Bi(x)\frac{dGi(x)}{dx}\right\} \tag{23}$$

$$\int_0^x Ai(t)dt = -\frac{2}{3} - \pi\left\{Hi(x)\frac{dAi(x)}{dx} - Ai(x)\frac{dHi(x)}{dx}\right\} \tag{24}$$

$$\int_0^x Bi(t)dt = -\pi\left\{Hi(x)\frac{dBi(x)}{dx} - Bi(x)\frac{dHi(x)}{dx}\right\}. \tag{25}$$

Now, by defining the following Wronskians:

$$W_1 = W(Ai(x), Gi(x)) = Ai(x)\frac{dGi(x)}{dx} - Gi(x)\frac{dAi(x)}{dx} \tag{26}$$

$$W_2 = W(Ai(x), Hi(x)) = Ai(x)\frac{dHi(x)}{dx} - Hi(x)\frac{dAi(x)}{dx} \tag{27}$$

$$W_3 = W(Bi(x), Gi(x)) = Bi(x)\frac{dGi(x)}{dx} - Gi(x)\frac{dBi(x)}{dx} \tag{28}$$

$$W_4 = W(Bi(x), Hi(x)) = Bi(x)\frac{dHi(x)}{dx} - Hi(x)\frac{dBi(x)}{dx} \tag{29}$$

we can write equations (22)-(25) in the following compact forms, respectively:

$$\int_0^x Ai(t)dt = \frac{1}{3} - \pi W_1 \tag{30}$$

$$\int_0^x Bi(t)dt = -\pi W_3 \tag{31}$$

$$\int_0^x Ai(t)dt = -\frac{2}{3} + \pi W_2 \tag{32}$$

$$\int_0^x Bi(t)dt = \pi W_4 . \tag{33}$$

Multiplying equation (23) by *Ai(x)* and (22) by *Bi(x)*, then subtracting the latter product from the former, and making use of the Wronskian of *Ai(x)* and *Bi(x)*, we obtain the following two expressions for *Ni(x):*

$$Ni(x) = Gi(x) - \frac{1}{3}Bi(x) \tag{34}$$

$$Ni(x) = \frac{2}{3}Bi(x) - Hi(x) . \tag{35}$$

The following relationship is obtained by using (21) in (34) or (35):

$$Ni(x) = \frac{2}{3}Gi(x) - \frac{1}{3}Hi(x) . \tag{36}$$

In addition, upon using (30)-(33) in (16), we obtain the following expressions for $Ni(x)$:

$$Ni(x) = \left(\pi W_1 - \frac{1}{3}\right)Bi(x) - \pi W_3 Ai(x) \tag{37}$$

$$Ni(x) = \left(\frac{2}{3} - \pi W_2\right)Bi(x) + \pi W_4 Ai(x) . \tag{38}$$

Now, using (11) and (12) in (36), we obtain the following integral representation for *Ni(x)*:

$$Ni(x) = \frac{2}{3\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt - \frac{1}{3\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt . \tag{39}$$

3.2. Derivatives of Ni(x) and their values at x=0:

First derivative of $Ni(x)$ is given by equation (17). We can write the second and third derivatives of $Ni(x)$ in the following equivalent forms, respectively:

$$\frac{d^2Ni(x)}{dx^2} = x\left\{Ai(x)\int_0^x Bi(t)dt - Bi(x)\int_0^x Ai(t)dt\right\} - W(Ai(x), Bi(x)) \tag{40}$$

or

$$\frac{d^2Ni(x)}{dx^2} = xNi(x) - W(Ai(x), Bi(x)) \tag{41}$$

and

$$\frac{d^3Ni(x)}{dx^3} = \left\{Ai(x) + x\frac{dAi(x)}{dx}\right\}\int_0^x Bi(t)dt - \left\{Bi(x) + x\frac{dBi(x)}{dx}\right\}\int_0^x Ai(t)dt \tag{42}$$

or

$$\frac{d^3Ni(x)}{dx^3} = Ni(x) + x\frac{dNi(x)}{dx} . \tag{43}$$

Higher derivatives of $Ni(x)$ involve $W(Ai(x), Bi(x)) = -\frac{d^2Ni(x)}{dx^2} = -\frac{1}{\pi}$ and the functions $Ni(x)$ and $\frac{dNi(x)}{dx}$. We develop the following iterative formula for computing the *n+1*[st] derivative of $Ni(x)$ with the knowledge of its *nth* derivative. Assuming that the *nth* derivative is of the form:

$$\frac{d^nNi(x)}{dx^n} = g(x)Ni(x) + h(x)\frac{dNi(x)}{dx} - p(x)W(Ai(x), Bi(x)) \tag{44}$$

where *g(x), h(x),* and *p(x)* are the coefficients of, $\frac{dNi(x)}{dx}$ and $W(Ai(x), Bi(x))$, respectively, that appear in the *nth* derivative, then the *n+1*[st] derivative is given by [2]:

$$\frac{d^{n+1}Ni(x)}{dx^{n+1}}=[\frac{dg(x)}{dx}+xh(x)]Ni(x)+[g(x)+\frac{dh(x)}{dx}]\frac{dNi(x)}{dx}-[h(x)+\frac{dp(x)}{dx}]W(Ai(x),Bi(x)). \quad (45)$$

Equation (45) takes the following form in terms of $Ai(x)$ and $Bi(x)$:

$$\frac{d^{n+1}Ni(x)}{dx^{n+1}}=\left[\{\frac{dg(x)}{dx}+xh(x)\}Ai(x)+\{g(x)+\frac{dh}{dx}\}\frac{dAi(x)}{dx}\right]\int_0^x Bi(t)dt$$
$$-\left[\{\frac{dg(x)}{dx}+xh(x)\}Bi(x)+\{g(x)+\frac{dh}{dx}\}\frac{dBi(x)}{dx}\right]\int_0^x Ai(t)dt-[h(x)+\frac{dp(x)}{dx}]W(Ai(x),Bi(x)). \quad (46)$$

The value of the *n+1*[st] derivative at $x=0$ can be obtained from either equation (45) or (46) and is written as:

$$\frac{d^{n+1}Ni(0)}{dx^{n+1}}=-\frac{1}{\pi}[h(0)+\frac{dp(0)}{dx}]. \quad (47)$$

Alternatively, we develop the following iterative formula for a more convenient way of computations:

$$\frac{d^{n+1}Ni(0)}{dx^{n+1}}=(n-1)\frac{d^{n-2}Ni(0)}{dx^{n-2}}; \quad \text{n=2,3,4,… .} \quad (48)$$

For ease of reference, we tabulate the derivatives of $Ni(x)$ and their values at $x=0$ in **Table 2**, below:

$Ni(x)$ **and its derivatives**	**Values at** $x=0$
$Ni(x)=Ai(x)\int_0^x Bi(t)dt-Bi(x)\int_0^x Ai(t)dt$	$Ni(0)=0$
$\frac{dNi(x)}{dx}=\frac{dAi}{dx}\int_0^x Bi(t)dt-\frac{dBi}{dx}\int_0^x Ai(t)dt$	$\frac{dNi(0)}{dx}=0$
$\frac{d^2Ni(x)}{dx^2}=xNi(x)-\frac{1}{\pi}$	$\frac{d^2Ni(0)}{dx^2}=-\frac{1}{\pi}$
$\frac{d^3Ni(x)}{dx^3}=x\frac{dNi(x)}{dx}+Ni(x)$	$\frac{d^3Ni(0)}{dx^3}=0$
$\frac{d^4Ni(x)}{dx^4}=2\frac{dNi(x)}{dx}+x^2Ni(x)-\frac{1}{\pi}x$	$\frac{d^4Ni(0)}{dx^4}=0$
$\frac{d^5Ni(x)}{dx^5}=x^2\frac{dNi(x)}{dx}+4xNi(x)-\frac{3}{\pi}$	$\frac{d^5Ni(0)}{dx^5}=-\frac{3}{\pi}$
$\frac{d^6Ni(x)}{dx^6}=6x\frac{dNi(x)}{dx}+[4+x^3]Ni(x)-\frac{1}{\pi}x^2$	$\frac{d^6Ni(0)}{dx^6}=0$
$\frac{d^7Ni(x)}{dx^7}=[10+x^3]\frac{dNi(x)}{dx}+9x^2Ni(x)-\frac{8}{\pi}x$	$\frac{d^7Ni(0)}{dx^7}=0$
$\frac{d^nNi(x)}{dx^n}=g(x)Ni(x)+h(x)\frac{dNi(x)}{dx}-\frac{1}{\pi}p(x)$	$\frac{d^nNi(0)}{dx^n}=-\frac{1}{\pi}p(0)$ or $\frac{d^nNi(0)}{dx^n}=(n-2)\frac{d^{n-3}Ni(0)}{dx^{n-3}}$; n=3,4,5,…
$\frac{d^{n+1}Ni(x)}{dx^{n+1}}=[\frac{dg(x)}{dx}+xh(x)]Ni(x)+[g(x)+\frac{dh(x)}{dx}]\frac{dNi(x)}{dx}$ $-[h(x)+\frac{dp(x)}{dx}]W(Ai(x),Bi(x)); n=2,3,4,...$	$\frac{d^{n+1}Ni(0)}{dx^{n+1}}=-\frac{1}{\pi}[h(0)+\frac{dp(0)}{dx}]$ or $\frac{d^{n+1}Ni(0)}{dx^{n+1}}=(n-1)\frac{d^{n-2}Ni(0)}{dx^{n-2}}$; n=2,3,4,…

Table2. $Ni(x)$ **And its derivatives and their values at** ***x = 0.***

3.3. Representations in terms of Bessel Functions

With the knowledge of the expressions of $Ai(x)$ and $Bi(x)$, their first derivatives and integrals in terms of Bessel's function, I_υ, J_υ, namely. [7,8]:

$$Ai(x)=\frac{\sqrt{x}}{3}\left[I_{-1/3}(\mu)-I_{1/3}(\mu)\right] \tag{49}$$

$$\frac{dAi(x)}{dx}=-\frac{x}{3}\left[I_{-2/3}(\mu)-I_{2/3}(\mu)\right] \tag{50}$$

$$Bi(x)=\sqrt{\frac{x}{3}}\left[I_{-1/3}(\mu)+I_{1/3}(\mu)\right] \tag{51}$$

$$\frac{dBi(x)}{dx}=\frac{x}{\sqrt{3}}\left[I_{-2/3}(\mu)+I_{2/3}(\mu)\right] \tag{52}$$

$$\int_0^x Ai(t)dt=\frac{1}{3}\int_0^\mu\left[I_{-1/3}(t)-I_{1/3}(t)\right]dt \tag{53}$$

$$\int_0^x Bi(t)dt=\frac{1}{\sqrt{3}}\int_0^\mu\left[I_{-1/3}(t)+I_{1/3}(t)\right]dt \tag{54}$$

where $\mu=\frac{2}{3}x^{3/2}$, we can express $Ni(x)$ and its derivative, defined in equations (16) and (17), in terms of Bessel's function, respectively, as

$$Ni(x)=\frac{2\sqrt{x}}{3\sqrt{3}}\left[I_{-1/3}(\mu)\right]\int_0^\mu\left[I_{1/3}(t)\right]dt-\frac{2\sqrt{x}}{3\sqrt{3}}\left[I_{1/3}(\mu)\right]\int_0^\mu\left[I_{-1/3}(t)\right]dt \tag{55}$$

$$\frac{dNi(x)}{dx}=\frac{2x}{3\sqrt{3}}\left[I_{2/3}(\mu)\right]\int_0^\mu\left[I_{1/3}(t)\right]dt-\frac{2x}{3\sqrt{3}}\left[I_{-2/3}(\mu)\right]\int_0^\mu\left[I_{-1/3}(t)\right]dt\,. \tag{56}$$

We can also express $Ni(x)$ and its derivative using the modified Bessel function, namely $K_\upsilon(t)=\frac{\pi}{2}\frac{I_{-\upsilon}(t)-I_\upsilon(t)}{\sin(\pi\upsilon)}$, and the Hankel function, namely $\overline{H}_\upsilon(t)=J_\upsilon(t)+iY_\upsilon(t)$, where $Y_\upsilon(t)$ is the Weber function, defined as: $Y_\upsilon(t)=\frac{J_\upsilon(t)-J_{-\upsilon}(t)}{\sin(\pi\upsilon)}$. This is accomplished by relying on the following expressions of Airy's functions and their derivatives in terms of Hankel and the modified Bessel functions, [7,8]:

$$Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}\left[K_{1/3}(\mu)\right] \tag{57}$$

$$\frac{dAi(x)}{dx}=-\frac{1}{\pi}\frac{x}{\sqrt{3}}\left[K_{2/3}(\mu)\right] \tag{58}$$

$$Bi(x)=\sqrt{\frac{x}{3}}\,\mathrm{Re}\left[e^{i\pi/6}\overline{H}_{1/3}(-i\mu)\right] \tag{59}$$

$$\frac{dBi(x)}{dx}=\frac{x}{\sqrt{3}}\,\mathrm{Re}\left[e^{i\pi/6}\overline{H}_{2/3}(-i\mu)\right]. \tag{60}$$

Using (57)-(60) in (16) and (17) we obtain, respectively

$$Ni(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}\left\{\left[K_{1/3}(\mu)\right]\int_0^x\sqrt{\frac{t}{3}}\,\mathrm{Re}\left[e^{i\pi/6}\overline{H}_{1/3}(-i\xi)\right]dt-\mathrm{Re}\left[e^{i\pi/6}\overline{H}_{1/3}(-i\mu)\right]\int_0^x\sqrt{\frac{t}{3}}\left[K_{1/3}(\xi)\right]dt\right\} \tag{61}$$

$$\frac{dNi(x)}{dx}=-\frac{1}{\pi}\frac{x}{\sqrt{3}}\left\{\left[K_{2/3}(\mu)\right]\int_0^x\sqrt{\frac{t}{3}}\,\mathrm{Re}\left[e^{i\pi/6}\overline{H}_{1/3}(-i\xi)\right]dt+\mathrm{Re}\left[e^{i\pi/6}\overline{H}_{2/3}(-i\mu)\right]\int_0^x\sqrt{\frac{t}{3}}\left[K_{1/3}(\xi)\right]dt\right\} \tag{62}$$

where $\xi = \frac{2}{3} t^{3/2}$.

3.4. Series Representations and Computation of $Ni(x)$

Computing and evaluating $Ni(x)$ are necessary in solving initial and boundary value problems involving the inhomogeneous Airy's equation, and entail evaluating Airy's functions. Typically, Airy's functions are expressed in terms of asymptotic or ascending series that provide approximations to these functions at given values of *x*. Since the $Ni(x)$ integral function is defined in terms of Airy's functions, we will rely on Airy's functions approximations to express $Ni(x)$ in terms of the asymptotic and ascending series for $Ai(x)$ and $Bi(x)$, and their derivatives and integrals.

3.4.1 Ascending Series Representation

Asymptotic series approximation of $Ni(x)$, valid for large *x*, has been discussed by Nield and Kuznetsov [1] who provided the following approximation and implemented it in their computations of flow over porous layers:

$$Ni(x) \approx -\frac{\exp(\frac{2}{3}x^{3/2})}{3\sqrt{\pi}x^{1/4}} \tag{63}$$

$$\int_0^x Ni(t)dt \approx -\frac{\exp(\frac{2}{3}x^{3/2})}{3\sqrt{\pi}x^{3/4}}. \tag{64}$$

3.4.2 Ascending Series Representation

For small values of *x*, we develop the following ascending series representation.
Letting

$$h_1 = Ai(0) = \frac{1}{\pi}\int_0^\infty \cos\frac{t^3}{3}dt \approx 0.3550280538878172 \tag{65}$$

$$h_2 = -\frac{dAi}{dx}(0) = \frac{1}{\pi}\int_0^\infty t\sin\frac{t^3}{3}dt \approx 0.2588194037928067 \tag{66}$$

$$F_1(x) = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \tag{67}$$

and

$$F_2(x) = \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \tag{68}$$

then

$$F_1'(x) = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} \tag{69}$$

$$F_2'(x) = \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \tag{70}$$

where $(b)_k$ is the Pochhammer symbol, defined as, [7]:

$$(b)_0 = 1 \tag{71}$$

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)(b+2)...(b+k-1); k > 0 \tag{72}$$

then the following representations can then be obtained, [7]:

$$Ai(x) = h_1 \frac{dF_1}{dx} - h_2 \frac{dF_2}{dx} \tag{73}$$

$$Bi(x) = \sqrt{3}h_1 \frac{dF_1}{dx} + \sqrt{3}h_2 \frac{dF_2}{dx} \tag{74}$$

$$\int_0^x A_i(t)dt = a_1 F_1(x) - a_2 F_2(x) \tag{75}$$

$$\int_0^x B_i(t)dt = \sqrt{3}a_1 F_1(x) + \sqrt{3}a_2 F_2(x) \tag{76}$$

Using (65)-(72) in (73)-(76) we obtain the following ascending series representations of Airy's functions and their integrals:

$$Ai(x) = h_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} - h_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \tag{77}$$

$$Bi(x) = \sqrt{3}h_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} + \sqrt{3}h_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \tag{78}$$

$$\int_0^x Ai(t)dt = h_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} - h_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \tag{79}$$

$$\int_0^x Bi(t)dt = \sqrt{3}h_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} + \sqrt{3}h_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!}. \tag{80}$$

In order to obtain an ascending series representation for $Ni(x)$, we use (65)-(72) in (16) to get either

$$Ni(x) = 2\sqrt{3}h_1 h_2 \left\{ F_2(x) \frac{dF_1}{dx} - F_1(x) \frac{dF_2}{dx} \right\} \tag{81}$$

or

$$Ni(x) = Ai(x)\left\{\sqrt{3}h_1 F_1(x) + \sqrt{3}h_2 F_2(x)\right\} - Bi(x)\left\{h_1 F_1(x) - h_2 F_2(x)\right\}. \tag{82}$$

Equations (81) and (82) have the following equivalent summation expressions, respectively

$$Ni(x) = 2\sqrt{3}h_1 h_2 \left[\begin{array}{l} \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} \right\} \left\{ \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\} \\ - \left\{ \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \end{array} \right] \tag{83}$$

$$\begin{aligned} Ni(x) = Ai(x) &\left\{ \sqrt{3}h_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} + \sqrt{3}h_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\} \\ - Bi(x) &\left\{ h_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} - h_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\} . \end{aligned} \tag{84}$$

Equation (84) can be expressed in the following form when we make use of the definition of Cauchy product:

$$N_i(x) = 2\sqrt{3}a_1 a_2 \sum_{k=0}^{\infty} 3^k \left(\sum_{l=0}^{k} \left(\frac{1}{3}\right)_l \left(\frac{2}{3}\right)_{k-l} \left(\frac{-3k+6l-1}{(3l+1)!(3(k-l)+2)!} \right) \right) x^{3k+2} \tag{85}$$

which we can integrate to get:

$$\int_0^x N_i(t)dt = 2\sqrt{3}a_1 a_2 \sum_{k=0}^{\infty} 3^k \left(\sum_{l=0}^{k} \left(\frac{1}{3}\right)_l \left(\frac{2}{3}\right)_{k-l} \left(\frac{-3k+6l-1}{(3l+1)!(3(k-l)+2)!} \right) \right) \frac{x^{3k+3}}{3k+3}. \tag{86}$$

In order to demonstrate utility of the asymptotic series expansions, above, we produced **Tables 3(a,b)** which contain values of $Ai(x)$, $Bi(x)$ and $Ni(x)$ for x = 1,2,3,…,10 (**Table 3(a)**), and for x = 0.1, 0.2, …, 1

(**Table 3(b)**). Computed values in **Table 3(b)** compare well with the values reported in [ ], with an agreement of at least seven significant digits.

Graph of $Ni(x)$ on the interval [0,1] is shown in Fig. 1, which shows its progressively decreasing behaviour with increasing *x*. Comparison of graphs of $Ai(x)$, $Bi(x)$ and $Ni(x)$ is shown in **Fig. 2** which demonstrates the opposite behaviours of $Bi(x)$ and $Ni(x)$. This behaviour is expected since an approximation to $Ni(x)$ is $-\frac{1}{3}Bi(x)$.

x	Ai(x)	Bi(x)	Ni(x)
1	0.1352924163	1.207423595	−0.1672560919
2	0.0349241297	3.298094995	−0.9304114376
3	0.006591136	14.03732897	−4.564880838
4	0.00095155	83.84707136	−27.86619072
5	0.0001081	657.7920428	−219.1991376
6	0.000009	6536.446114	−2178.767510
7	−0.00002	80327.79058	−26776.86426
8	−0.0002	$1.199586009 \times 10^{6}$	$-4.000836962 \times 10^{5}$
9	−0.007	$2.147286890 \times 10^{7}$	$-7.193803430 \times 10^{6}$
10	−0.1	$4.556411543 \times 10^{8}$	$-1.623380420 \times 10^{8}$

Table 3(a). Computed Values of $Ai(x)$, $Bi(x)$ and $Ni(x)$ Using Ascending Series

x	Ai(x)	Bi(x)	Ni(x)
0.1	0.3292031298	0.6598616903	−0.001591629009
0.2	0.3037031543	0.7054642032	−0.006368744579
0.3	0.2788064818	0.7524855851	−0.01434329156
0.4	0.2547423541	0.8017730001	−0.02554637147
0.5	0.2316936062	0.8542770430	−0.04003797119
0.6	0.2098000612	0.9110633416	−0.05791696555
0.7	0.1891624001	0.9733286565	−0.07933159847
0.8	0.1698463170	1.042422172	−0.1044907107
0.9	0.1518868034	1.119872814	−0.1336760520
1.0	0.1352924163	1.207423595	−0.1672560919

Table 3(b). Computed Values of $Ai(x)$, $Bi(x)$ and $Ni(x)$ Using Ascending Series

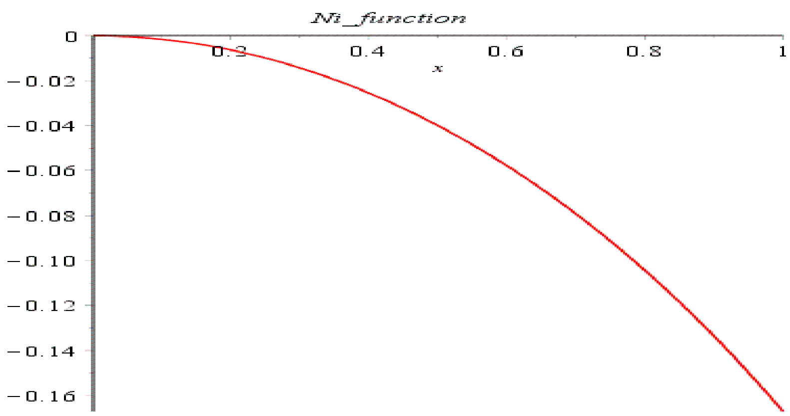

Fig. 1. $Ni(x)$ for $0 \le x \le 1$

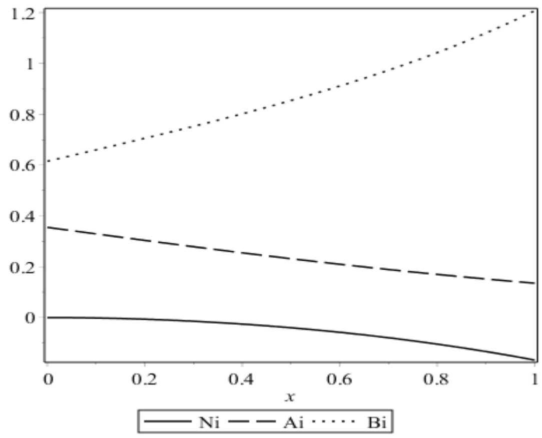

Fig. 2. $Ai(x), Bi(x), Ni(x)$ for $0 \le x \le 1$

3.4.3 Representation of the Solution to the Initial Value Problem

Using the ascending and asymptotic series representations above, we evaluate and plot solution (18) for the following data: $\alpha = 1, \beta = 2$ and *R* = 1, -1, $\mp \frac{1}{\pi}$. These are presented in **Figs. 3(a-d)**. All of these figures show the discrepancy between the asymptotic and ascending series solutions. Since the interval chosen is [0,1] and the asymptotic series solution is valid for *x* >>1, it is expected that the asymptotic series solution is inaccurate as compared to the ascending series solution which is valid for small vales of *x*. This behaviour is independent of the choice of the forcing function *R*.

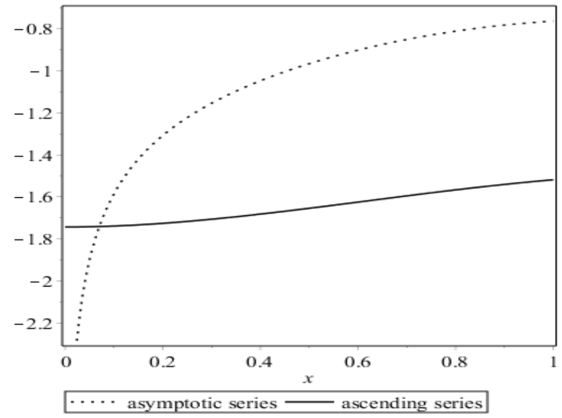

Fig. 3(a) Ascending and Asymptotic Series Solutions for *R*=1

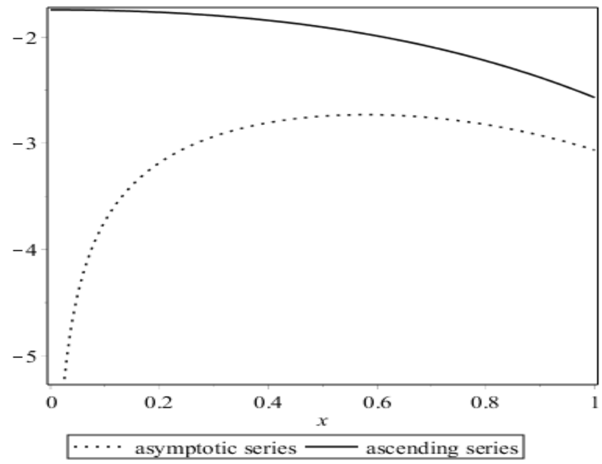

Fig. 3(b) Ascending and Asymptotic Series Solutions for *R*=-1

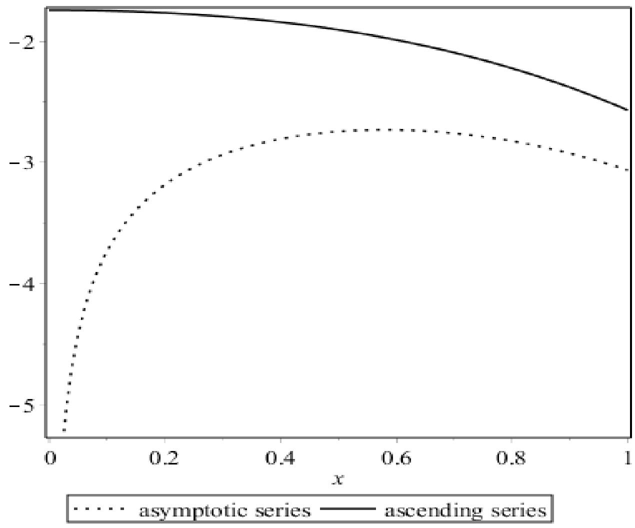

Fig. 3(c) Ascending and Asymptotic Series Solutions for $R = \frac{1}{\pi}$

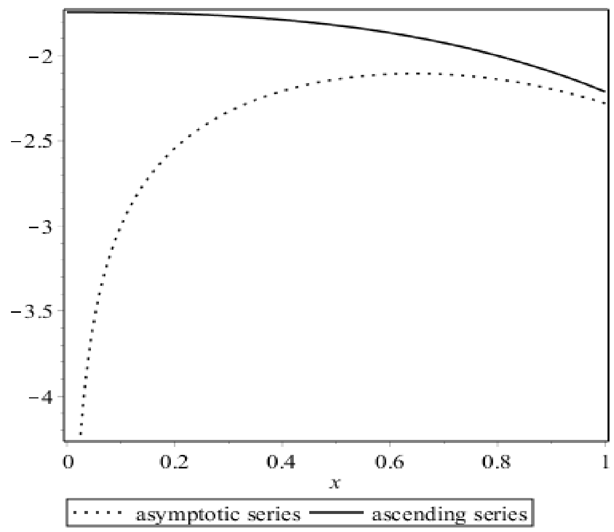

Fig. 3(d) Ascending and Asymptotic Series Solutions for $R = -\frac{1}{\pi}$

IV. Conclusion

In this work, we provided further properties and representations of the Nield-Kuznetsov function. In particular, we expressed $Ni(x)$ in terms of Bessel and Hankel functions, and provided an ascending series expression for its evaluation. An initial value problem was solved, and the solution was computed using both asymptotic and ascending series for comparison. Suitability of the ascending series representation is emphasized.

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