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Nield- Kuznetsov Functions of the First- and Second Kind

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Abstract: *The Nield-Kuznetsov integral functions of the first- and second-kind are introduced and classified. Each kind is sub-classified as standard, generalized or parametric, depending on the source of each function. Power series expressions are derived and Tables of values of functions of the first-kind are generated by evaluating the derived series.*

Keywords: *Nield-Kuznetsov, Integral functions, Weber and Airy ODE.*

I. INTRODUCTION

In their elegant analysis of the transition layer, Nield and Kuznetsov [1] demonstrated the usefulness of Brinkman's equation with variable permeability. The ease by which they obtained an exact solution to the governing equations with the help of Airy's functions initiated revival of special functions that are of great utility in the study of flow through variable permeability media. This includes the recent attention by some authors to the use of generalized Airy's and Weber's equations in the study of flow through and over porous layers, [2-4]. In the process of obtaining a solution to the resulting boundary value problem, Nield and Kuznetsov [1] introduced a new integral function, $Ni(x)$, referred to as the Nield-Kuznetsov function, [5], which represents the basis for solving Airy's inhomogeneous differential equation with initial or boundary conditions and a constant or variable forcing function. This function continues to be abundantly studied (*cf.* [4-8] and the references therein) due to its useful properties, its various representations, and extensions of its concept to related functions arising in other differential equations. For instance, since Airy's equation is a special case of the generalized Airy's equation that was introduced by Swanson and Headley, [9], it seems natural to extend the Nield-Kuznetsov function to a generalized function suitable for solving the inhomogeneous generalized Airy's equation. A generalized Nield-Kuznetsov function was recently introduced to serve this purpose, [3], and has been successfully implemented in the solution of flow through composite porous layers. The same applies to Weber's inhomogeneous equation where a parametric Nield-Kuznetsov function was recently introduced, [4]. In the current work, we introduce and classify the Nield-Kuznetsov functions, present an application in which they arise, define the functions for variable forcing functions, and develop series representation for the functions. The series representations will them be used to generate preliminary Tables of values of the Nield-Kuznetsov functions.

II. POISEUILLE FLOW THROUGH A POROUS CHANNEL

In case of Poiseuille flow through a porous channel bounded by porous plates, as shown in **Fig. 1**, below, Brinkman's equation with variable permeability, [3], reduces to the following ordinary differential equation:

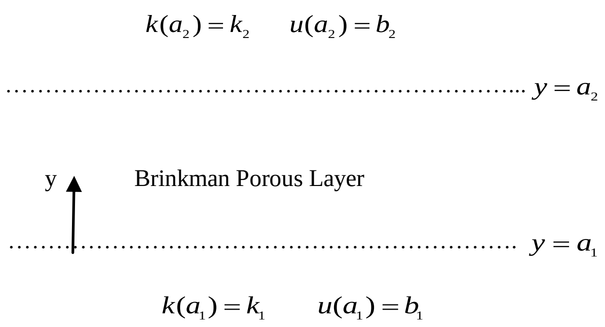

Fig. 1. Representative Sketch

$$\frac{d^2u}{dy^2} - \frac{\mu}{\mu_{eff}k(y)}u = \kappa \qquad ...(1)$$

where $\kappa = p_x / \mu_{eff}$, $u = u(y)$ is the tangential velocity, $k(y)$ is the permeability that varies as a function of the transverse direction to the flow, p_x is the constant driving pressure gradient, μ is the base viscosity of the fluid and μ_{eff} is the effective viscosity of the fluid traversing the porous layer. In the configuration of **Fig. 1**, equation (1) is to be solved for $u(y)$ subject to the conditions:

$$u(a_1) = b_1 \qquad \ldots (2)$$

$$u(a_2) = b_2 \qquad \ldots (3)$$

with $k(y)$ varying according to either of the forms:

$$k(a_1) = k_1 \le k(y) \le k(a_2) = k_2 \qquad \ldots (4)$$

$$k(a_1) = k_1 \ge k(y) \ge k(a_2) = k_2 \qquad \ldots (5)$$

where b_1, b_2 , k_1 and k_2 are specified non-negative real numbers.

The boundary value problem composed of solving (1) subject to conditions (2) and (3), with the permeability distribution satisfying (4) or (5) is in general what is encountered in transition layer analysis. Equation (1) is a linear, inhomogeneous ordinary differential equation with variable coefficients that depend on the distribution of the variable permeability $k(y)$. We will select permeability distributions that reduce equation (1) to a special form of an equation with solution expressible in terms of integral functions. This approach provides us with further insights into the nature of the resulting special functions, and an expanded range of applicability of those functions.

III. PERMEABILITY DISTRIBUTIONS

In what follows, we will discuss three forms of the variable permeability distributions that result in well-known and classical differential equations whose solutions are integral functions. **Table 1** below lists three forms of variable permeability function and the resulting variable coefficient differential equation that equation (1) reduces to.

Form #	$k(y)$	Resulting differential equation
1	$\dfrac{\mu}{\mu_{eff}\, y}$	$\dfrac{d^2u}{dy^2} - yu = \kappa$ (Airy's equation)
2	$\dfrac{\mu}{\mu_{eff}\, y^n}$	$\dfrac{d^2u}{dy^2} - y^n u = \kappa$ (Generalized Airy's equation)
3	$\dfrac{4\mu}{\mu_{eff}(4a - y^2)}$	$\dfrac{d^2u}{dy^2} + (\dfrac{y^2}{4} - a)u = \kappa$; $a \in \Re$ (Weber's equation).

Table 1. Variable Permeability Functions and the Resulting Differential Equations.

Solutions to the homogeneous parts of the equations in **Table 1** are well-documented in the literature, and given in terms of linear combinations of two linearly independent functions, u_1, u_2 , given in **Table 2**, together with their Wronskians, $\varpi(u_1, u_2)$.

Differential Equation	u_1	u_2	$\varpi(u_1, u_2)$
Airy's equation	$Ai(y)$	$Bi(y)$	$1/\pi$
Generalized Airy's equation	$A_n(y)$	$B_n(y)$	$2(\sqrt{m}/\pi)\sin(m\pi)$; where $m = 1/(n+2)$

Weber equation	$w(a,y)$	$w(a,-y)$	1

Table 2. Linearly independent functions satisfying the homogeneous equations

The Wronskians in **Table 2** are defined:

$$\varpi(u_1,u_2)=\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}=u_1u_2'-u_2u_1'. \quad \text{...(6)}$$

The functions $A_n(y)$ and $B_n(y)$ are the generalized Airy's functions, of the first and second kinds, respectively, defined in terms of modified Bessel's function, I_m, of the first kind of order $m=1/(n+2)$, by [9]:

$$A_n(y)=m\sqrt{y}[I_{-m}(\mu)-I_m(\mu)] \quad \text{...(7)}$$

$$B_n(y)=\sqrt{my}[I_{-m}(\mu)+I_m(\mu)] \quad \text{...(8)}$$

wherein $\mu=2my^{\frac{1}{2m}}=\dfrac{2}{n+2}y^{(n+2)/2}$, Γ is the gamma function, and

$$I_m(\mu)=\sum_{r=1}^{\infty}\frac{1}{r!\Gamma(r+m+1)}\left(\frac{\mu}{2}\right)^{2r+m}. \quad \text{...(9)}$$

The standard Airy's functions of the first and second kinds, $Ai(y)$ and $Bi(y)$, respectively, are a special case of (7) and (8), obtained by taking n=1, m=1/3.

Solutions to Weber's differential equations are the linearly independent, numerically satisfactory pair of functions $W(a,\mp y)$, known as parabolic cylinder functions, [10-15], valid where y and *a* are real numbers.

IV. PARTICULAR SOLUTIONS

4.1. The case of constant forcing function:

Particular solutions are obtained by the method of variation of parameters and take the forms given in **Table 3** together with the general solutions, where c_1, c_2, c_{1n}, c_{2n}, c_{1w} and c_{2w} are arbitrary constants:

Differential Equation	**Particular solution**	**General solution**
Airy's eqtn.	$u_p=-\pi\kappa Ni(y)$	$u=c_1Ai(y)+c_2Bi(y)-\pi\kappa Ni(y)$
Generalized Airy's eqtn	$(u_p)_n=-\kappa\dfrac{\pi}{2\sqrt{m}\sin(m\pi)}N_n(y)$	$u_n=c_{1n}A_n(y)+c_{2n}B_n(y)-\dfrac{\kappa\pi}{2\sqrt{m}\sin(m\pi)}N_n(y)$
Weber eqtn.	$(u_p)_w=-\kappa N_w(a,y)$	$u_w=c_{1w}W(a,y)+c_{2w}W(a,-y)-\kappa N_w(a,y)$

Table 3: Particular and general solutions of the inhomogeneous equations: Forcing function κ

We point out that in **Table 3**, when n=1, solutions to Airy's equation are recovered from solution to generalized Airy's equation. The functions $Ni(y)$, $N_n(y)$ and $N_w(a,y)$ are the Nield-Kuznetsov integral functions of the ***first kind***, defined in **Table 4**, and arise when the forcing functions in the inhomogeneous equations are constant. Later in this work we define the Nield-Kuznetsov functions of the *second kind*, which arise when the forcing function is a variable function of y.

4.2. The case of variable forcing function:

When the constant κ in **Table 1** replaced with a variable function $f(y)$, solutions to the homogeneous parts of Airy's, generalized Airy's and Weber's equations are not affected by $f(y)$, and are obtained by taking linear combinations of u_1, u_2 of **Table 2**. Particular solutions are influenced by $f(y)$ and can be obtained by the method of variation of parameters as follows. **Table 5** gives the particular solutions, written in terms of the Nield-Kuznetsov functions of the ***second kind***, given in **Table 6**, where we assumed that $f(y) = F'(y)$.

Integral Function	**Integral Form and First Derivative**
Standard Nield-Kuznetsov function of the first kind	$Ni(y) = Ai(y)\int_0^y Bi(t)dt - Bi(y)\int_0^y Ai(t)dt$ $N'i(y) = A'i(y)\int_0^y Bi(t)dt - B'i(y)\int_0^y Ai(t)dt$
Generalized Nield-Kuznetsov function of the first kind	$N_n(y) = A_n(y)\int_0^y B_n(t)dt - B_n(y)\int_0^y A_n(t)dt$ $N_n'(y) = A_n'(y)\int_0^y B_n(t)dt - B_n'(y)\int_0^y A_n(t)dt$
Parametric Nield-Kuznetsov function of the first kind	$N_w(a,y) = W(a,y)\int_0^y W(a,-t)dt - W(a,-y)\int_0^y W(a,t)dt$ $N_w'(a,y) = W'(a,y)\int_0^y W(a,-t)dt + W'(a,-y)\int_0^y W(a,t)dt$

Table 4. The Nield-Kuznetsov Functions of the First Kind

Differential Equation	**Particular solution**	**General solution**
Airy's eqtn	$u_p = -\pi Ki(y)$	$u = c_1 Ai(y) + c_2 Bi(y) - \pi Ki(y)$
Generalized Airy's eqtn	$(u_p)_n = -\frac{\pi}{2\sqrt{m}\sin(m\pi)} K_n(y)$	$u_n = c_{1n} A_n(y) + c_{2n} B_n(y) - \frac{\pi}{2\sqrt{m}\sin(m\pi)} K_n(y)$
Weber eqtn	$(u_p)_w = -K_w(a,y)$	$u_w = c_{1w} W(a,y) + c_{2w} W(a,-y) - K_w(a,y)$

Table 5: Particular and general solutions of the inhomogeneous equations: forcing function $f(y)$

The Nield-Kuznetsov Functions of the second kind and their first derivatives can thus be defined as given in **Table 6**, below:

Integral Function	Integral Form and First Derivative
Standard Nield-Kuznetsov function of the second kind	$Ki(y) = -[Ai(y)\int_0^y F(t)B'i(t)dt - Bi(y)\int_0^y F(t)A'i(t)dt]$ $K'i(y) = -\left[A'i(y)\int_0^y F(t)B'i(t)dt - B'i(y)\int_0^y F(t)A'i(t)dt + \frac{F(y)}{\pi}\right]$
Generalized Nield-Kuznetsov function of the second kind	$K_n(y) = -[A_n(y)\int_0^y F(t)B'_n(t)dt - B_n(y)\int_0^y F(t)A'_n(t)dt]$ $K'_n(y) = -\left[A'_n(y)\int_0^y F(t)B'_n(t)dt - B'_n(y)\int_0^y F(t)A'_n(t)dt + \frac{2\sqrt{m}\sin(m\pi)}{\pi}F(y)\right]$
Parametric Nield-Kuznetsov function of the second kind	$K_w(a,y) = W(a,-y)\int_0^y F(t)W'(a,t)dt + W(a,y)\int_0^y F(t)W'(a,-t)dt$ $K'_w(a,y) = W'(a,y)\int_0^y F(t)W'(a,-t)dt - W'(a,-y)\int_0^y F(t)W'(a,t)dt - F(y)$

Table 6. The Nield-Kuznetsov Functions of the First Kind

V. VALUES AT ZERO

The generalized Airy's functions have been shown to have the following power series expansions [9]:

$$A_n(y) = \alpha_n g_{n1}(y) - \beta_n g_{n2}(y) \quad \text{…(10)}$$

$$B_n(y) = [\alpha_n g_{n1}(y) + \beta_n g_{n2}(y)]/\sqrt{m} \quad \text{…(11)}$$

$$\alpha_n = (m)^{1-m}/\Gamma(1-m) \quad \text{…(12)}$$

$$\beta_n = (m)^m/\Gamma(m) \quad \text{…(13)}$$

$$g_{n1}(y) = 1 + \sum_{j=1}^{\infty} m^{2j} \prod_{p=1}^{j} \frac{y^{j(n+2)}}{p(p-m)} \quad \text{…(14)}$$

$$g_{n2}(y) = y[1 + \sum_{j=1}^{\infty} m^{2j} \prod_{p=1}^{j} \frac{y^{j(n+2)}}{p(p+m)}]. \quad \text{…(15)}$$

Equations (10)-(15) are evaluated at $y = 0$ to generate values in **Table 7**, below. For the values of Airy's functions and first derivatives at $y = 0$, we use n=1 and m=1/3 in (10)-(15). In addition, values of $W(a,0)$ and $W'(a,0)$ have been reported in the literature [15], and take the following expressions, respectively:

$$W(a,0) = \frac{1}{(2)^{3/4}} \left| \frac{\Gamma(\frac{1}{4} + \frac{ia}{2})}{\Gamma(\frac{3}{4} + \frac{ia}{2})} \right|^{1/2} \quad \text{…(16)}$$

$$W'(a,0) = -\frac{1}{(2)^{1/4}}\left|\frac{\Gamma(\frac{3}{4}+\frac{ia}{2})}{\Gamma(\frac{1}{4}+\frac{ia}{2})}\right|^{1/2}. \quad \ldots(17)$$

Values of the Nield-Kuznetsov functions at *y = 0* are obtained from their definitions in **Tables 4** and **6**. These values are given in **Table 8**, below.

Airy's Functions	**Generalized Airy's Functions**	**Weber's Functions**
$Ai(0) = \frac{(1/3)^{2/3}}{\Gamma(2/3)}$	$A_n(0) = \frac{(m)^{1-m}}{\Gamma(1-m)}$	$W(a,0) = \frac{1}{(2)^{3/4}}\left\|\frac{\Gamma(\frac{1}{4}+\frac{ia}{2})}{\Gamma(\frac{3}{4}+\frac{ia}{2})}\right\|^{1/2}$
$Bi(0) = \frac{(1/3)^{1/6}}{\Gamma(2/3)}$	$B_n(0) = \frac{(m)^{1/2-m}}{\Gamma(1-m)}$	$W(0,0) = \frac{1}{(2)^{3/4}}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^{1/2}$
$A'(0) = -\frac{(1/3)^{1/3}}{\Gamma(1/3)}$	$A_n'(0) = -\frac{(m)^{m}}{\Gamma(m)}$	$W'(a,0) = -\frac{1}{(2)^{1/4}}\left\|\frac{\Gamma(\frac{3}{4}+\frac{ia}{2})}{\Gamma(\frac{1}{4}+\frac{ia}{2})}\right\|^{1/2}$
$B'(0) = \frac{(3)^{1/6}}{\Gamma(1/3)}$	$B_n'(0) = \frac{(m)^{m-1/2}}{\Gamma(m)}$	$W'(0,0) = -\frac{1}{(2)^{1/4}}\left(\frac{\Gamma(3/4)}{\Gamma(1/4)}\right)^{1/2}$

Table 7. Values of Airy's, Generalized Airy's and Weber Functions and Derivatives at Zero

Nield-Kuznetsov Functions of the First Kind	**Nield-Kuznetsov Functions of the Second Kind**
$Ni(0) = 0$; $N'i(0) = 0$	$Ki(0) = 0$; $K'i(0) = -F(0)/\pi$
$N_n(0) = 0$; $N_n'(0) = 0$	$K_n(0) = 0$; $K_n'(0) = -2\sqrt{m}\sin(m\pi)F(0)/\pi$
$N_w(a,0) = 0$; $N_w'(a,0) = 0$	$K_w(a,0) = 0$; $K_w'(a,0) = -F(0)$

Table 8. Values of the Nield-Kuznetsov Functions and Derivatives at Zero

VI. POWER SERIES REPRESENTATION

Evaluations of the Nield-Kuznetsov functions at given values of the independent variable are essential for solving initial and boundary value problems. We derive power series expressions for these functions.

6.1. Series Expressions for the Standard Nield-Kuznetsov Functions:

The following ascending series expressions for $Ni(y)$ and $Ki(y)$ have been derived elsewhere (*cf.* [7]):

$$N_i(y)=2\sqrt{3}a_1a_2\left[\begin{array}{l}\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k y^{3k}}{(3k)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k y^{3k+2}}{(3k+2)!}\right\}\\-\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k y^{3k+1}}{(3k+1)!}\right\}\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k y^{3k+1}}{(3k+1)!}\right\}\end{array}\right] \quad \ldots(18)$$

$$\begin{aligned}K_i(y)=&\ f(y)N_i(y)-\left[\sqrt{3}a_2A_i(y)+a_2B_i(y)\right]\int_0^y f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt\\&-\left[\sqrt{3}a_1A_i(y)-a_1B_i(y)\right]\int_0^y f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k t^{3k}}{(3k)!}\right\}dt\end{aligned} \quad \ldots(19)$$

$$A_i(y)=a_1\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k y^{3k}}{(3k)!}-a_2\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k y^{3k+1}}{(3k+1)!} \quad \ldots(20)$$

$$B_i(y)=\sqrt{3}a_1\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k y^{3k}}{(3k)!}+\sqrt{3}a_2\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k y^{3k+1}}{(3k+1)!} \quad \ldots(21)$$

$$a_1=Ai(0)=\frac{1}{\pi}\int_0^{\infty}\cos\frac{t^3}{3}dt \quad \ldots(22)$$

$$a_2=-\frac{dAi}{dx}(0)=\frac{1}{\pi}\int_0^{\infty}t\sin\frac{t^3}{3}dt\,. \quad \ldots(23)$$

Series (19) involves $f(y)$ and $Ni(y)$ which must be determined before $K_i(y)$ is determined. By writing $f(y)=F'(y)$, the more convenient definition of $Ki(y)$, given in **Table 6**, is obtained. Following the procedure described in [7], we obtain the following series representation for $Ki(y)$:

$$Ki(y)=-\left(\begin{array}{l}2\sqrt{3}a_1a_2\left(\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k\frac{3^k y^{3k}}{(3k)!}\right)\left[\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\int_0^y F(t)\frac{3^k t^{3k}}{(3k)!}dt\right]\\-2\sqrt{3}a_1a_2\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k\frac{3^k y^{3k+1}}{(3k+1)!}\left[\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)_k\int_0^y F(t)\frac{3^k t^{3k-1}}{(3k-1)!}dt\right]\end{array}\right). \quad \ldots(24)$$

6.2. Series Expressions for the Generalized Nield-Kuznetsov Functions:

The generalized Airy's functions are given in equations (10)-(15), above. We use these equations to derive series expressions for $N_n(y)$ and $K_n(y)$ defined in Tables (4) and (6), as follows. Writing (14) and (15) as:

$$g_{n1}(y)=1+\sum_{j=1}^{\infty}m^{2j}\left(y^{j^2/m}\right)\prod_{p=1}^{j}\frac{1}{p(p-m)} \quad \ldots(25)$$

$$g_{n2}(y)=y+\sum_{j=1}^{\infty}m^{2j}\left(y^{1+j^2/m}\right)\prod_{p=1}^{j}\frac{1}{p(p+m)}] \quad \ldots(26)$$

and upong differentiating and integrating (25) and (26) we obtain:

$$g'_{n1}(y) = \sum_{j=1}^{\infty} m^{2j-1} j^2 \left(y^{-1+j^2/m}\right) \prod_{p=1}^{j} \frac{1}{p(p-m)} \qquad \ldots(27)$$

$$g'_{n2}(y) = 1 + \sum_{j=1}^{\infty} m^{2j} (1 + \frac{j^2}{m}) \left(y^{j^2/m}\right) \prod_{p=1}^{j} \frac{1}{p(p+m)}] \qquad \ldots(28)$$

$$\int_0^y [g_{n1}(t)]dt = y + \sum_{j=1}^{\infty} m^{2j} \frac{1}{(1+j^2/m)} \left(y^{1+j^2/m}\right) \prod_{p=1}^{j} \frac{1}{p(p-m)} \qquad \ldots(29)$$

$$\int_0^y [g_{n2}(t)]dt = \frac{y^2}{2} + \sum_{j=1}^{\infty} m^{2j} \frac{1}{(2+j^2/m)} \left(y^{2+j^2/m}\right) \prod_{p=1}^{j} \frac{1}{p(p+m)}]. \qquad \ldots(30)$$

Using (25)-(30) in the expressions for $N_n(y)$ and $K_n(y)$ in Tables 4() and (6), we obtain:

$$N_n(y) = \frac{2}{\sqrt{m}} \alpha_n \beta_n \left[g_{n1}(y) \int_0^y [g_{n2}(t)]dt - g_{n2}(y) \int_0^y [g_{n1}(t)]dt \right] \qquad \ldots(31)$$

$$K_n(y) = \frac{-2}{\sqrt{m}} \alpha_n \beta_n \left[g_{n1}(y) \int_0^y [F(t) g'_{n2}(t)]dt - g_{n2}(y) \int_0^y [F(t) g'_{n1}(t)]dt \right]. \qquad \ldots(32)$$

6.3. Series Expressions for the Parametric Nield-Kuznetsov Functions:

The following expressions, developed in [10-15], for the Weber functions $W(a,y)$ and $W(a,-y)$:

$$W(a,y) = W(a,0) \sum_{n=0}^{\infty} \rho_n(a) \frac{y^{2n}}{(2n)!} + W'(a,0) \sum_{n=0}^{\infty} \delta_n(a) \frac{y^{2n+1}}{(2n+1)!} \qquad \ldots(33)$$

$$W(a,-y) = W(a,0) \sum_{n=0}^{\infty} \rho_n(a) \frac{y^{2n}}{(2n)!} - W'(a,0) \sum_{n=0}^{\infty} \delta_n(a) \frac{y^{2n+1}}{(2n+1)!} \qquad \ldots(34)$$

$$W'(a,y) = W(a,0) \sum_{n=1}^{\infty} \rho_n(a) \frac{y^{2n-1}}{(2n-1)!} + W'(a,0) \sum_{n=1}^{\infty} \delta_n(a) \frac{y^{2n}}{(2n)!} \qquad \ldots(35)$$

$$W'(a,-y) = W'(a,0) \sum_{n=1}^{\infty} \delta_n(a) \frac{y^{2n}}{(2n)!} - W(a,0) \sum_{n=1}^{\infty} \rho_n(a) \frac{y^{2n-1}}{(2n-1)!} \qquad \ldots(36)$$

$$\rho_{n+2} = a\rho_{n+1} - \frac{1}{2}(n+1)(2n+1)\rho_n \qquad \ldots(37)$$

$$\delta_{n+2} = a\delta_{n+1} - \frac{1}{2}(n+1)(2n+3)\delta_n \qquad \ldots(38)$$

$$\rho_0(a) = 1;\ \rho_1(a) = a;\ \delta_0(a) = 1;\ \delta_1(a) = a \qquad \ldots(39)$$

and $W(a,0)$, $W'(a,0)$ are as given in **Table 7**. Using equations (33)-(39) in the definitions of $N_w(a,y)$ and $K_w(a,y)$, given in Tables (4) and (6), we obtain the following series expressions:

$$N_w(a,y) = 2W(a,0)W'(a,0)\left[\begin{array}{l}\left\{\sum_{n=0}^{\infty}\delta_n(a)\frac{y^{2n+1}}{(2n+1)!}\right\}\left\{\sum_{n=0}^{\infty}\rho_n(a)\frac{y^{2n+1}}{(2n+1)!}\right\} \\ -\left\{\sum_{n=0}^{\infty}\rho_n(a)\frac{y^{2n}}{(2n)!}\right\}\left\{\sum_{n=0}^{\infty}\delta_n(a)\frac{y^{2n+2}}{(2n+2)!}\right\}\end{array}\right] \quad \ldots(40)$$

$$K_w(a,y) = 2W'(a,0)W(a,0)[\sum_{n=0}^{\infty}\rho_n(a)\frac{y^{2n}}{(2n)!}]\int_0^y[\sum_{n=1}^{\infty}\delta_n(a)F(t)\frac{t^{2n}}{(2n)!}]dt\,. \quad \ldots(41)$$

VII. Tables of Values of the Nield-Kuznetsov Functions of the First Kind

Power series (18), (31) and (40) have been evaluated using *Maple* with full computational accuracy, that is, without setting an upper limit on the number of terms used. The following Tables of values are generated for $0 \le y \le 1$ with a step of 0.1.

y	$Ni(y)$
0	0
0.1	-0.001591629009
0.2	-0.006368744579
0.3	-0.01434329156
0.4	-0.02554637147
0.5	-0.04003797119
0.6	-0.05791696555
0.7	-0.07933159847
0.8	-0.1044907107
0.9	-0.1336760520
1	- 0.1672560919

Table 9. Values of $Ni(y)$ using series (18)

y	$N_1(y) = Ni(y)$	$N_2(y)$	$N_3(y)$
0	0	0	0
0.1	-0.001591629008	-0.001125399147	- 0.0008367272500
0.2	-0.006368744771	-0.004501821668	- 0.003346933706
0.3	-0.01434329669	-0.01013129344	- 0.007530979168
0.4	-0.02554642263	-0.01802169498	- 0.01339089697
0.5	-0.04003826894	-0.02819352974	- 0.02093374202
0.6	-0.05791816009	-0.04068944470	- 0.03017795748
0.7	-0.07933502829	-0.05558659595	- 0.04116378680
0.8	-0.1044967395	-0.07301204525	- 0.05396883072
0.9	-0.1336725243	-0.09316151260	- 0.06873025908
1	- 0.1671679498	- 0.1163218192	- 0.08567790804

Table 10(a). Values of $N_n(y)$ using series (31)

y	$N_4(y)$	$N_5(y)$	$N_{10}(y)$
0	0	0	0
0.1	- 0.0006497473468	- 0.0005220047831	- 0.0002378240305
0.2	- 0.002598992311	- 0.002088019499	- 0.0009512961219
0.3	- 0.005847802133	- 0.004698057311	- 0.002140416280
0.4	- 0.01039671775	- 0.008352266574	- 0.003805184840
0.5	- 0.01624821576	- 0.01305153560	- 0.005945608733
0.6	- 0.02341039345	- 0.01879947876	- 0.008561767496

0.7	- 0.03190451911	- 0.02560749306	- 0.01165426375
0.8	- 0.04177858228	- 0.03350563491	- 0.01522648504
0.9	- 0.05312966759	- 0.04256347834	- 0.01929364193
1	- 0.06614082173	- 0.05292832891	- 0.02391327336

Table 10(b). Values of $N_n(y)$ using series (31)

y	$N_w(a,y);a=0$	$N_w(a,y);a=0.5$	$N_w(a,y);a=1$
0	0	0	0
0.1	- 0.004999995827	- 0.005002079512	- 0.005004163888
0.2	- 0.01999973332	- 0.02003308857	- 0.02006648827
0.3	- 0.04499696254	- 0.04516595734	- 0.04533545954
0.4	- 0.07998293445	- 0.08051760658	- 0.08105513505
0.5	- 0.1249349069	- 0.1262419177	- 0.1275598486
0.6	- 0.1798056698	- 0.1825197255	- 0.1852664744
0.7	- 0.2445101224	- 0.2495456854	- 0.2546639319
0.8	- 0.3189089751	- 0.3275118455	- 0.3362995397
0.9	- 0.4027896942	- 0.4165877536	- 0.4307617641
1	- 0.4958448910	- 0.5168969487	- 0.5386588510

Table 11. Values of $N_w(a,y)$ using series (40)

VIII. CONCLUSION

In this work we introduced, redefined, and classified three Nield-Kuznetsov functions of the first-kind and three of the second-kind, and showed a physical situation in which they arise. The Power series expressions were derived for all six functions and preliminary **Tables** of values for the Nield-Kuznetsov functions of the first-kind were obtained. This work sets the stage for further studies of the properties, applications, and the development of more Nield-Kuznetsov integral functions.

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