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Differential Operators Invariant under Lorentz Transformation

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Abstract: *Using differential forms, we obtain, in an entirely new way, expressions for differential operators: gradient, divergence and curl which are invariant under Lorentz transformation. The expression for $curlF$ of a vector field F contains $curl\overline{F}$ as a term, where $\overline{F}$ is a spatial part of F and the other term in it is related to the direction of time. This is but natural in 4-dimensional space-time structure. We have not found such a simple form for the expression of $curlF$ in the existing literature on Special Relativity. In addition we give expressions in spherical polar coordinates for the above mentioned differential operators. In this article we have used the standard Lorentz transformation in which the moving frame moves along one of the coordinate axes of the stationary frame. Considering a more general form of Lorentz transformation in which the moving frame moves with uniform velocity in an arbitrary direction, the expressions of the differential operators were obtained earlier [2]. But the computations involved there were too heavy and complicated for an interested reader (especially if he belongs to a faculty of physical sciences). Computations in this article are very few in comparison and the new way of derivation may inspire further investigation in the field of Special Relativity.*

In order to understand the computations given in this article, all that one needs to know is that the product of differentials dx,dy etc. is skew symmetric and it is indicated by inserting a wedge between them. Thus $dx \wedge dy = -dy \wedge dx$. Another property is that of Hodge star operator [3]. About this, the details are given where necessary.

I. Preliminaries

Minkowski space (M,η) is a globally hyperbolic Lorentz manifold with a flat metric η. Consider a coordinate system (x^0,x^1,x^2,x^3) in the underlying manifold E^4 with $x^0 = ct$. We assume that the velocity of light c is unity, so that $x^0 = t$ has the unit of length. The x^1,x^2,x^3 are space coordinates. The basis vectors $e_\alpha = \dfrac{\partial}{\partial x^\alpha}$, $\alpha = 0,1,2,3$ satisfy:

(1.1)
$$\eta(e_\alpha,e_\beta) = \eta_{\alpha\beta} = -1 \quad \text{if } \alpha = \beta = 0$$
$$= 1 \quad \text{if } \alpha, \beta = 1,2,3$$
$$= 0 \quad \text{if } \alpha \neq \beta$$

Standard LT is a transformation from one system of coordinates x^α in a Lorentz frame Σ into another system $x^{\beta'}$ in a Lorentz frame Σ' which is moving with uniform velocity v along x^1-axis. The transformation equations are given by [4], [5],

$$x^{\beta'} = \wedge^{\beta'}_{\alpha}\, x^\alpha,\ x^\alpha = \wedge^{\alpha}_{\beta'}\, x^{\beta'}$$

(1.2)

where we have followed Einstein summation convention for repeated indices and the coefficient matrices are constant matrices which satisfy:

(1.3)
$$a) \qquad \wedge^{\alpha'}_{\gamma}\ \wedge^{\beta'}_{\delta}\ \eta_{\alpha'\beta'} = \eta_{\gamma\delta}$$
$$b) \qquad \wedge^{\alpha}_{\gamma'}\ \wedge^{\beta}_{\delta'}\ \eta_{\alpha\beta} = \eta_{\gamma'\delta'}$$
$$c) \qquad \wedge^{\alpha'}_{\beta}\ \wedge^{\beta}_{\gamma'} = \delta^{\alpha'}_{\gamma'}$$
$$d) \qquad \wedge^{\alpha}_{\beta'}\ \wedge^{\beta'}_{\gamma} = \delta^{\alpha}_{\gamma}$$

Note that $\eta_{\alpha'\beta'}$ are defined as in (1.1) by setting:

$$\eta_{\alpha'\beta'} = \eta(e_{\alpha'},e_{\beta'})$$

with

$$e_{\alpha'} = \frac{\partial}{\partial x^{\alpha'}}.$$

Since the coefficient matrices in (1.2) are constants we have:

(1.4)
$$dx^{\beta'} = \wedge_{\alpha}^{\beta'} dx^{\alpha}$$
$$dx^{\alpha} = \wedge_{\beta'}^{\alpha} dx^{\beta'}.$$

Hence

$$\eta_{\alpha'\beta'}\ dx^{\alpha'} dx^{\beta'} = \eta_{\alpha'\beta'} \wedge_{\gamma}^{\alpha'} \wedge_{\delta}^{\beta'} dx^{\gamma} dx^{\delta} = \eta_{\gamma\delta}\ dx^{\gamma} dx^{\delta}.$$

(1.5)

We get

(1.6)
$$d\tau^2 = -\eta_{\alpha\beta}\ dx^{\alpha} dx^{\beta}$$
$$= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Then (1.5) shows that $d\tau$ is invariant under a Lorentz transformation. τ is called the proper time . The coordinate expression of a world line $X(\tau)$ of a particle is given by:

$$X(\tau) = x^{\alpha} e_{\alpha} = x^{\beta'} e_{\beta'}$$

(1.7)

Since the basis vectors do not depend on the parameter τ , we have $dx^{\alpha} e_{\alpha} = dx^{\beta'} e_{\beta'}$. Using (1.4) we obtain:

(1.8) $\quad dx^{\beta'} e_{\beta'} = \wedge_{\gamma}^{\beta'} dx^{\gamma} e_{\beta'} = dx^{\gamma} e_{\gamma}$, so $\wedge_{\gamma}^{\beta'} e_{\beta'} = e_{\gamma}$

The inner product in Σ is given by:

(1.9) $\quad < e_{\alpha}\ , e_{\beta} > \ = e_{\alpha} \cdot e_{\beta} = \eta_{\alpha\beta}$

and that in Σ' is given by:

$$< e_{\alpha'}\ , e_{\beta'} > \ = e_{\alpha'} \cdot e_{\beta'} = \eta_{\alpha'\beta'}\ ,$$

$\eta_{\alpha\beta}$ and $\eta_{\alpha'\beta'}$ are related to each other as in (1.3-a,b).

Invariance of volume element dV and that of star operator $*dX$ under LT.

Explicit expressions for $\wedge_{\beta}^{\alpha'}$ and $\wedge_{\beta'}^{\alpha}$ are as given below, [4]:

(1.10) $$\left\|\wedge_{\beta}^{\alpha'}\right\| = \left\|\begin{matrix} \Gamma & -\Gamma v & 0 & 0 \\ -\Gamma v & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\| \quad \text{and} \quad \left\|\wedge_{\beta'}^{\alpha}\right\| = \left\|\begin{matrix} \Gamma & \Gamma v & 0 & 0 \\ \Gamma v & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\|$$

Using these and (1.2) we have:

(1.11) $\quad x^0 = \Gamma x^{0'} + \Gamma v x^{1'}\ ,\ x^1 = \Gamma v x^{0'} + \Gamma x^{1'},\ x^2 = x^{2'},\ \ x^3 = x^{3'}, x^0 = ct\ , \Gamma = \dfrac{1}{\sqrt{1-v^2}}$

where, c is assumed to be 1. We show that the volume element $dV = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0$ is invariant under LT. We have from (1.11):

(1.12)
$$dx^1 \wedge dx^0 = \left(\Gamma v dx^{0'} + \Gamma dx^{1'}\right) \wedge \left(\Gamma dx^{0'} + \Gamma v dx^{1'}\right)$$
$$= \Gamma^2 v^2 dx^{0'} \wedge dx^{1'} + \Gamma^2 dx^{1'} \wedge dx^{0'}$$
$$= \left(-\Gamma^2 v^2 + \Gamma^2\right) dx^{1'} \wedge dx^{0'}$$
$$= dx^{1'} \wedge dx^{0'}, \quad \text{since} \quad \Gamma^2\left(1-v^2\right) = 1$$

Since $x^2 = x^{2'},\ \ x^3 = x^{3'},\qquad dx^2 \wedge dx^3 = dx^{2'} \wedge dx^{3'}$

and
$$dV = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0$$
$$= dx^2 \wedge dx^3 \wedge dx^1 \wedge dx^0 \quad \text{using skew symmetric property of wedge product}$$

$$= dx^{2'} \wedge dx^{3'} \wedge dx^{1'} \wedge dx^{0'} \quad \text{using (1.12)}$$
$$= dx^{1'} \wedge dx^{2'} \wedge dx^{3'} \wedge dx^{0'} = dV'$$

that is, the volume element is invariant under LT.

Now we show that $*dX$ is invariant, where $*$ is Hodge's star operator. For details of $*$ operator see [3].

Let $X' = x^{\alpha'} e_{\alpha'}$ be the position vector of a point $X = x^{\alpha} e_{\alpha}$ which has undergone LT.

Then

(1.13) $\quad *dX = *dx^{\alpha} e_{\alpha}$, $\quad *dX' = *dx^{\alpha'} e_{\alpha'}$

We have [3]

$$*dX' = dx^{2'} \wedge dx^{3'} \wedge dx^{0'} e_{1'} + dx^{3'} \wedge dx^{1'} \wedge dx^{0'} e_{2'}$$
$$+ dx^{1'} \wedge dx^{2'} \wedge dx^{0'} e_{3'} + dx^{1'} \wedge dx^{2'} \wedge dx^{3'} e_{0'}$$

From (1.2),(1.8) and (1.10) we have:

(1.14) $\quad x^{0'} = \Gamma x^0 - \Gamma v x^1$, $\quad x^{1'} = -\Gamma v x^0 + \Gamma x^1$, $\quad x^{2'} = x^2$, $\quad x^{3'} = x^3$

$$e_{0'} = \Gamma e_0 + \Gamma v e_1, \quad e_{1'} = \Gamma v e_0 + \Gamma e_1$$

and

(1.15) $\quad e_{2'} = \wedge^{\alpha}_{2'} e_{\alpha} = \wedge^{2}_{2'} e_2 = e_2$, since $\quad \wedge^{\alpha}_{2'} = 0, \quad \alpha = 0,1,3$ and $\wedge^{2}_{2'} = 1$

Similarly it is easy to see that $e_{3'} = e_3$. From (1.12), (1.14) and (1.15) we have:

(1.16)
$$dx^{3'} \wedge dx^{1'} \wedge dx^{0'} e_{2'} = dx^3 \wedge dx^1 \wedge dx^0 e_2 \quad \text{and}$$
$$dx^{1'} \wedge dx^{2'} \wedge dx^{0'} e_{3'} = dx^1 \wedge dx^2 \wedge dx^0 e_3$$

Now:

a)
$$dx^{2'} \wedge dx^{3'} \wedge dx^{0'} e_{1'} = dx^2 \wedge dx^3 \wedge \left(\Gamma dx^0 - \Gamma v dx^1\right)\left(\Gamma v e_0 + \Gamma e_1\right)$$
$$= \Gamma^2 v dx^2 \wedge dx^3 \wedge dx^0 e_0 - \Gamma^2 v^2 dx^2 \wedge dx^3 \wedge dx^1 e_0$$
$$+ \Gamma^2 dx^2 \wedge dx^3 \wedge dx^0 e_1 - \Gamma^2 v dx^2 \wedge dx^3 \wedge dx^1 e_1$$

and

b)
$$dx^{1'} \wedge dx^{2'} \wedge dx^{3'} e_{0'} = \left(-\Gamma v dx^0 + \Gamma dx^1\right) \wedge dx^2 \wedge dx^3 \left(\Gamma e_0 + \Gamma v e_1\right)$$
$$= -\Gamma^2 v dx^0 \wedge dx^2 \wedge dx^3 e_0 - \Gamma^2 v^2 dx^0 \wedge dx^2 \wedge dx^3 e_1$$
$$+ \Gamma^2 dx^1 \wedge dx^2 \wedge dx^3 e_0 + \Gamma^2 v dx^1 \wedge dx^2 \wedge dx^3 e_1$$
$$= -\Gamma^2 v dx^2 \wedge dx^3 \wedge dx^0 e_0 - \Gamma^2 v^2 dx^2 \wedge dx^3 \wedge dx^0 e_1$$
$$+ \Gamma^2 dx^2 \wedge dx^3 \wedge dx^1 e_0 + \Gamma^2 v dx^2 \wedge dx^3 \wedge dx^1 e_1$$

Adding the above expressions in a) and b) and observing that $\Gamma^2(1 - v^2) = 1$ we get:

(1.17) $\quad dx^{2'} \wedge dx^{3'} \wedge dx^{0'} e_{1'} + dx^{1'} \wedge dx^{2'} \wedge dx^{3'} e_{0'} = dx^2 \wedge dx^3 \wedge dx^0 e_1 + dx^1 \wedge dx^2 \wedge dx^3 e_0$

From (1.16) and (1.17) we get $*dX' = *dX$. Thus $*dX$ is invariant under LT.

II. Differential operators in Minkowski space

Before we derive expressions for the differential operators, we give a few steps necessary for that purpose. From (1.13) we have

(2.1) $\quad dX \cdot \wedge * dX = dx^{\alpha} e_{\alpha} \cdot \wedge * dx^{\beta} e_{\beta} = dx^{\alpha} \wedge * dx^{\beta} < e_{\beta}, e_{\beta} > = dx^{\alpha} \wedge * dx^{\beta} \eta_{\alpha\beta}$

Since

$$\eta_{00} dx^0 \wedge * dx^0 = -dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0 = dV$$

or

$$dx^0 \wedge * dx^0 = \eta^{00} dV$$

and

$$\eta_{11} dx^1 \wedge * dx^1 = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0 = \eta^{11} dV \text{ etc,}$$

We have

(2.2) $dx^{\alpha} \wedge *dx^{\beta} = \eta^{\alpha\beta} dV$

We have shown that both dV and $*dX$ are invariant with respect to LT. We use these in the study of differential operators.

Lemma 2.1 Let f be a differentiable real valued function and F, a differentiable vectorfield on (M,η) Then

(2.3) $a)\ df \wedge *dX = \eta^{\alpha\beta} \frac{\partial f}{\partial x^{\alpha}} e_{\beta}\ dV = \left(-\frac{\partial f}{\partial x^0} e_0 + \frac{\partial f}{\partial x^1} e_1 + \frac{\partial f}{\partial x^2} e_2 + \frac{\partial f}{\partial x^3} e_3 \right) dV = (grad f) dV$

$b)\ dF \cdot \wedge * dX = \frac{\partial F^{\mu}}{\partial x^{\mu}} dV = \left(\frac{\partial F^0}{\partial x^0} + \frac{\partial F^1}{\partial x^1} + \frac{\partial F^2}{\partial x^2} + \frac{\partial F^3}{\partial x^3} \right) dV = (\mathrm{div} F) dV$

$c)\ d(grad f) \cdot \wedge * dX = \eta^{\alpha\gamma} \frac{\partial^2 f}{\partial x^{\alpha} \partial x^{\gamma}} dV = \left(-\frac{\partial^2 f}{\partial x^{0^2}} + \frac{\partial^2 f}{\partial x^{1^2}} + \frac{\partial^2 f}{\partial x^{2^2}} + \frac{\partial^2 f}{\partial x^{3^2}} \right) dV = \Box f\ dV$

$d)\ curl F = \frac{\partial F^{\alpha}}{\partial x^{\beta}} \eta^{\beta\gamma}\ e_{\gamma} \times e_{\alpha} = \left(grad F^0 + \frac{\partial \overline{F}}{\partial x^0} \right) \times e_0 + \left(curl \overline{F} \right)_3$

where $F = F^0 e_0 + \sum_{i=1}^{3} F^i e_i = F^0 e_0 + \overline{F}$

$\Box f$ is known as d'Alembertian of f [5] and $\cdot$ indicates dot product between vectors and $\times$ indicates cross product.

Proof: a)

$$df \wedge *dX = \frac{\partial f}{\partial x^{\alpha}} dx^{\alpha} \wedge *dx^{\beta} e_{\beta}$$

$$= \eta^{\alpha\beta} \frac{\partial f}{\partial x^{\alpha}}\ e_{\beta}\ dV \qquad \text{by making use of (2.2)}$$

$$= (grad f) dV$$

If we use primed coordinate system, we obtain

$$df \wedge *dX' = \eta^{\alpha'\beta'} \frac{\partial f}{\partial x^{\alpha'}} e_{\beta'}\ dV' = (grad f)' dV'$$

Using the invariance of dV and $*dX$ we get

$$df \wedge *dX' = (grad f)' dV' = df \wedge *dX = (grad f) dV$$

Since $dV = dV'$ we have $grad f = (grad f)'$

b) Let $F = F^{\alpha} e_{\alpha}$ be a differentiable vector field in (M,η). Then $dF = dF^{\alpha} e_{\alpha}$ and

$$dF \wedge \cdot * dX = \frac{\partial F^{\mu}}{\partial x^{\beta}} dx^{\beta} \wedge *dx^{\alpha} \langle e_{\mu}, e_{\alpha} \rangle$$

$$= \frac{\partial F^{\mu}}{\partial x^{\beta}} \eta^{\beta\alpha}\ \eta_{\mu\alpha}\ dV, \quad since \quad < e_{\mu}, e_{\alpha} >= \eta_{\mu\alpha}$$

$$= \frac{\partial F^{\mu}}{\partial x^{\mu}} dV \qquad since \quad \eta^{\beta\alpha} \eta_{\mu\alpha} = \delta^{\beta}_{\mu}$$

$$= (div F) dV$$

where $\cdot$ indicates dot product between two vectors. Following the procedure used in (a) above, it is easy to show that *divF* does not depend on any particular frame.

Further we have

c) $$d(grad f) \cdot \wedge * dX = \eta^{\alpha\beta} d\left(\frac{\partial f}{\partial x^{\alpha}} \right) e_{\beta} \cdot \wedge * dx^{\mu} e_{\mu}$$

$$= \eta^{\alpha\beta}\eta_{\beta\mu}\frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma} dx^\gamma \wedge * dx^\mu$$

$$= \eta^{\gamma\mu}\frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma}\delta^\alpha_\mu dV, \qquad since \quad \eta^{\alpha\beta}\eta_{\beta\mu} = \delta^\alpha_\mu$$

$$= \eta^{\alpha\gamma}\frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma} dV$$

$$= \Box f \ dV$$

Since c) is a special case of b), it is clear that $\Box f$ also does not depend on any particular frame.
To prove d) some details are given below: □

Definition 2.1 Let F be a differentiable vectorfield on (M,η). $curlF$ is given by

(2.4) $$(curlF)dV = -dF \wedge \times * dX$$

where the cross indicates the vector product between vectors.

Lemma: 2.2

We have

(2.5) $$curlF = \frac{\partial F^\alpha}{\partial x^\beta}\eta^{\beta\gamma} \quad \mathrm{e}_\gamma \times e_\alpha$$

Proof

$$(curlF)dV = -dF \wedge \times * dX$$

$$= -\frac{\partial F^\alpha}{\partial x^\beta} dx^\beta \mathrm{e}_\alpha \wedge \times * dx^\gamma \ e_\gamma$$

$$= \frac{\partial F^\alpha}{\partial x^\beta}\eta^{\beta\gamma} \quad \mathrm{e}_\gamma \times \ e_\alpha \ dV$$

from which the result follows. □

Invariance of *curlF* **:**

If we use primed coordinate system, then since

$$F^{\alpha'} = F^\alpha \wedge^{\alpha'}_\alpha, \qquad \eta^{\beta'\gamma'} = \eta^{\beta\gamma} \wedge^{\beta'}_\beta \wedge^{\gamma'}_\gamma$$

$$\frac{\partial F^{\alpha'}}{\partial x^{\beta'}} = \frac{\partial F^\alpha}{\partial x^\mu} \wedge^{\alpha'}_\alpha \wedge^\mu_{\beta'}, \qquad e_{\gamma'} \times e_{\alpha'} = \wedge^\gamma_{\gamma'} \wedge^\alpha_{\alpha'} e_\gamma \times e_\alpha$$

Using (2.5) we get

$$\left(curlF\right)' dV' = \eta^{\beta'\gamma'}\frac{\partial F^{\alpha'}}{\partial x^{\beta'}} e_{\gamma'} \times e_{\alpha'} dV'$$

$$= \eta^{\beta\gamma} \wedge^{\beta'}_\beta \wedge^{\gamma'}_\gamma \frac{\partial F^\alpha}{\partial x^\mu} \wedge^{\alpha'}_\alpha \wedge^\mu_{\beta'} e_{\gamma'} \times e_{\alpha'} dV'$$

$$= \eta^{\beta\gamma}\delta^\mu_\beta \frac{\partial F^\alpha}{\partial x^\mu} e_\gamma \times e_\alpha dV$$, since $dV' = dV$ and $\wedge^{\alpha'}_\alpha e_{\alpha'} = e_\alpha$, $\wedge^{\gamma'}_\gamma e_{\gamma'} = e_\gamma$, using (1.8)

$$= \eta^{\beta\gamma}\frac{\partial F^\alpha}{\partial x^\beta} e_\gamma \times e_\alpha dV$$

$$= \left(curlF\right) dV$$

or

$$\left(curlF\right)' = curlF$$

This establishes the invariance, that is the expression for $curlF$ does not depend on a particular Lorentz frame.

Corollary 2.3 We have

$$curl(gradf) = 0$$

for every differentiable real valued function f defined on (M,η).

Proof

$$curl(gradf) = \frac{\partial}{\partial x^{\beta}}(\eta^{\alpha\delta}\frac{\partial f}{\partial x^{\delta}})\ \eta^{\beta\gamma}\ e_{\gamma}\times e_{\alpha}$$

$$= \eta^{\alpha\delta}\eta^{\beta\gamma}\frac{\partial^2 f}{\partial x^{\beta}\partial x^{\delta}}\ e_{\gamma}\times e_{\alpha}$$

Since

$$\frac{\partial^2 f}{\partial x^{\beta}\partial x^{\delta}} = \frac{\partial^2 f}{\partial x^{\delta}\partial x^{\beta}}$$

we have

$$curl(gradf) = \frac{1}{2}\left(\eta^{\alpha\delta}\eta^{\beta\gamma} + \eta^{\alpha\beta}\eta^{\delta\gamma}\right)\frac{\partial^2 f}{\partial x^{\beta}\partial x^{\delta}}\ e_{\gamma}\times e_{\alpha} = 0$$ ∎

Corollary 2.4

Let $\overline{F} = \sum_{j=1}^{3} F^{j} e_{j}$ be the space component of a vectorfield F on a (M,η).

Then

(2.6) $$curlF = \left(gradF^{0} + \frac{\partial \overline{F}}{\partial x^{0}}\right)\times e_0 + (curl\overline{F})_3$$

where $(curl\overline{F})_3$ denotes the curl of Euclidean vectorfield $\overline{F}$ in E^3.

Proof

Let $F = F^0 e_0 + F^j e_j = F^0 e_0 + \overline{F}$

$$curlF = \frac{\partial F^0}{\partial x^{\beta}}\eta^{\beta\gamma}\quad e_{\gamma}\times e_0 + \frac{\partial F^j}{\partial x^{\beta}}\eta^{\beta\gamma}\quad e_{\gamma}\times e_j \qquad j=1,2,3.$$

$$= (gradF^0)\times e_0 + \frac{\partial F^j}{\partial x^0}e_j\times e_0 + \frac{\partial F^j}{\partial x^k}\eta^{k\gamma}\quad e_{\gamma}\times e_j$$

$$= \left(gradF^0 + \frac{\partial \overline{F}}{\partial x^0}\right)\times e_0 + \frac{\partial F^j}{\partial x^k}\eta^{kl}e_l\times e_j \quad \text{since } \eta^{k0}=0, \quad k=1,2,3$$

$$= \left(gradF^0 + \frac{\partial \overline{F}}{\partial x^0}\right)\times e_0 + (curl\overline{F})_3$$ ∎

Note 2.1 $(curl\overline{F})_3 = \eta^{kl}\frac{\partial F^j}{\partial x^k}e_l\times e_j$

$$= \eta^{11}\frac{\partial F^j}{\partial x^1}e_1\times e_j + \eta^{22}\frac{\partial F^j}{\partial x^2}e_2\times e_j + \eta^{33}\frac{\partial F^j}{\partial x^3}e_3\times e_j$$

$$= \frac{\partial F^2}{\partial x^1}e_1\times e_2 + \frac{\partial F^3}{\partial x^1}e_1\times e_3 + \frac{\partial F^1}{\partial x^2}e_2\times e_1 + \frac{\partial F^3}{\partial x^2}e_2\times e_3 + \frac{\partial F^1}{\partial x^3}e_3\times e_1 + \frac{\partial F^2}{\partial x^3}e_3\times e_2$$

$$= \left(\frac{\partial F^2}{\partial x^1} - \frac{\partial F^1}{\partial x^2}\right)e_1\times e_2 + \left(\frac{\partial F^3}{\partial x^2} - \frac{\partial F^2}{\partial x^3}\right)e_2\times e_3 + \left(\frac{\partial F^1}{\partial x^3} - \frac{\partial F^3}{\partial x^1}\right)e_3\times e_1$$

This is the usual expression for $curl\overline{F}$ in E^3, and

(2.7) $$\left(gradF^0 + \frac{\partial \overline{F}}{\partial x^0}\right)\times e_0 = \left(\sum_{i=1}^{3}\frac{\partial F^0}{\partial x^i}e_i + \frac{\partial F^i}{\partial x^0}e_i\right)\times e_0$$

Remark 2.1.

Since e_0 is in the direction of time the pattern $e_1 \times e_2 = e_3$ etc does not apply to the vector product of e_i with e_0. This, perhaps, is the reason why the coefficient of $e_i \wedge e_0$ is not skew-symmetric.

Expressions for *gradf*, *divF* and *CurlF* in terms of spherical polar coordinates.

$$a) \quad gradf = -\frac{\partial f}{\partial x^0}e_0 + \frac{\partial f}{\partial r}e_r + \frac{1}{r}\frac{\partial f}{\partial \theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}e_\phi$$

$$b) \quad \text{div}F = \frac{\partial F^0}{\partial x^0} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F^\theta) + \frac{1}{r^2\sin^2\theta}\frac{\partial F^\phi}{\partial\phi}$$

(2.8) $\quad c) \quad \Box f =$ d'Alembertian of f

$$= -\frac{\partial^2 f}{\partial x^{0^2}} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$d) \quad curlF = \left[(gradF^0) + \frac{\partial\overline{F}}{\partial x^0}\right]\times e_0 + \left[\frac{1}{r}\frac{\partial}{\partial r}(rF^\theta) - \frac{1}{r}\frac{\partial F^r}{\partial\theta}\right]e_r\times e_\theta$$

$$+ \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(F^\phi\sin\theta) - \frac{\partial F^\theta}{\partial\phi}\right)e_\theta\times e_\phi + \left(\frac{1}{r\sin\theta}\frac{\partial F^r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rF^\phi)\right)e_\phi\times e_r$$

where

$X = x^0\, e_0 + r\sin\theta\cos\phi\, e_1 + r\sin\theta\sin\phi\, e_2 + r\cos\theta\, e_3$ and

$F = F^0 e_0 + F^r e_r + F^\theta e_\theta + F^\phi e_\phi$, $e_r = \sin\theta\cos\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\theta\, e_3$,

$e_\theta = \cos\theta\cos\phi\, e_1 + \cos\theta\sin\phi\, e_2 - \sin\theta\, e_3$ and $e_\phi = -\sin\phi\, e_1 + \cos\phi\, e_2$

The expression for $\Box f$ given here may be compared with the formula for d'Alembertian given in [1] on p.76. For details of derivation of these formulas see [2].

The motivation for using this new method of deriving expressions for differential operators was from such a derivation available in 3-dimensional Euclidean space. This is briefly indicated in the appendix.

Appendix: Differential operators in 3-dimensional space

Let $X = x^i e_i$ be the position vector of a point in E^3. Then $*dX = *dx^i e_i$ where $*$ is Hodge's operator.

$$*dx^1 = dx^2 \wedge dx^3,\ *dx^2 = dx^3 \wedge dx^1,\ *dx^3 = dx^1 \wedge dx^2$$

$dV = dx^1 \wedge *dx^1 \ldots$ etc., where dV is the volume element.

Let f be a real valued function on E^3 and $F = (F^1, F^2, F^3) = F^i e_i$ be a vector valued function. Then

$$df \wedge *dX = \frac{\partial f}{\partial x^i}dx^i \wedge *dx^j e_j\ ,\ dx^i \wedge *dx^j = \delta^{ij}dV$$

$$= \left(\frac{\partial f}{\partial x^1}e_1 + \frac{\partial f}{\partial x^2}e_2 + \frac{\partial f}{\partial x^3}e_3\right)dV = (gradf)dV$$

$$dF \cdot\wedge * dX = \frac{\partial F^i}{\partial x^k}dx^k \wedge *dx^l e_i \cdot e_l$$

$$= \delta_{il}\frac{\partial F^i}{\partial x^k}dx^k \wedge *dx^l$$

$$= \delta_{il}\delta^{kl}\frac{\partial F^i}{\partial x^k}dV = \frac{\partial F^i}{\partial x^i}\cdot dV = (divF)dV$$

$$dF \wedge \times *dX = \frac{\partial F^i}{\partial x^j}dx^j \wedge *dx^k e_i \times e_k$$

$$= \delta^{jk}dV\frac{\partial F^i}{\partial x^j}e_i \times e_k$$

$$= \frac{\partial F^1}{\partial x^2}\delta^{22} e_1 \times e_2 dV + \frac{\partial F^2}{\partial x^1}\delta^{11} e_2 \times e_1 dV + \ldots$$

$$= \left(\frac{\partial F^1}{\partial x^2} - \frac{\partial F^2}{\partial x^1} \right) e_1 \times e_2 dV + \ldots$$

$$= -curlF \; dV$$

In these, $*dX$ and dV depend on a coordinate system and are not invariant under a transformation of coordinate axes. This does not apply to df and dF. Under Lorentz transformation it has been possible to show that both dV and $*dX$ are invariant. This has helped us to define differential operators in an invariant way.

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