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Portfolio selection under different risk measures

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***Abstract*:** *This paper reviewed and compared four different risk measures which are applied in portfolio selection problem, and analyzed the portfolio selection models when distribution of asset returns are not given. Specifically, EVT approach is used to analyzed the portfolio selection problem.*
keywords: *Portfolio selection, safty-first, VaR, CVaR, EVT*

I. Introduction

Portfolio theory deals with the question of how to find an optimal distribution of the wealth among various assets. Markowitz provides a fundamental basis for portfolio selection in a single period. Analytical expression of the mean-variance efficient frontier in single-period portfolio selection was derived in Markowitz(1952) and Merton(1972). Recently, a multi-period portfolio selection problem has been studied. This problem is more interesting as investors always invest their wealth in multi periods instead of only one period. Literatures in this problem has considered of maximizing expected utility function of the terminal wealth. Li and Ng (2000)have derived the analytical formulation of the frontier of the multi-period portfolio selection by embedding the assets-only multi-period mean-variance problem into a large tractable problem.

Mean-Variance model is widely used in the practice of fund management. It is used for asset allocation to determine the basic policy of fund management as well as the management of individual funds including portfolio construction and risk control, etc. In the mean-variance framework, risk is defined in terms of possible variation of expected portfolio returns. The focus on standard deviation as appropriate measure for risk implies that investors weigh the probability of negative returns equally against positive returns. However, in the investment environment, various financial instruments with non-symmetric return distribution exist, such as options and bonds. In addition, recent statistical studies revealed that not all stocks follow normal distribution. So it is highly unlikely that the perception of investors to downside risk faced on investments is the same as the perception to the upward potential. Therefore some approaches have been taken to incorporate downside risk into the asset allocation model.

Safety-first approach proposed by Roy(1952) is one idea of considering downside risk in portfolio selection. Roy define the shortfall constraint such that the probability that the value of the portfolio falling below a specified disaster level is limited to a specified disaster probability. Also some literatures extend the safety-first portfolio selection model to multi-period case. Recently, an analytical solution is achieved for the multi-period safety-first formulation in Li and Ng (1998).

Risk measures such as *VaR*, *C aR* were also constructed in order to measure the downside risk of asymmetric return distributions. R. Tyrrell Rockafellar and Stanislav Uryasev (1999) provide an approach to optimizing a portfolio to reduce risk with *CV aR* as the risk measure. To measure the risk by *VaR* or *C aR*, we need to know the distribution of return, which we usually do not exactly know. Therefore, extreme value theory (*EVT*) was introduced. *EVT* provides a firm theoretical foundation on which we can build statistical models describing extreme events without knowing the distribution of returns.

In this paper, we firstly introduced the risk measures. In section 2, the risk measures are used to build the portfolio selection model. These models are compared in section 3. Section 4 will gives some methodology to handle these models. Finally, we will give the conclusion.

1. Risk Measures

1.1 Risk measures with given distribution function

Given a random variable *X* (e.g. the uncertain gain of invested risky asset) with density function $f(x)$ and continuous cumulative distribution function $F(x)$. μ is the mean of *X*. The risk measures are given as follows: (1) Variance $\sigma^2 := \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$; (2)Value-at-Risk (*VaR*) which is the maximum expected loss on an investment over a specified horizon given some confidence level $1-\alpha$, $VaR_\alpha := \max\{-x | F(x) \le \alpha\} = -F^{-1}(\alpha)$; (3) Conditional Value-at-Risk(*CVaR*) which measures the potential size of the loss exceeding *VaR*, $CVaR_\alpha := -E(x|x \le -CVaR_\alpha) = -\frac{1}{\alpha}\int_{-\infty}^{-VaR_\alpha} xf(x)dx$.

1.2 Risk measures without given distribution function

1.2.1 Extreme Value Theory (EV T)

Let $\xi_1, \xi_2, \cdots$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) and write M_n for the maximum of the first *n*, i.e. $M_n = \max\{\xi_1, \xi_2, \cdots \xi_n\}$.
Extreme value theory models the maximum of a random variable and tell us what the limiting distributions are. It states that a nondegenerate asymptotic distribution of M_n must belong to one of just three possible general families, regardless of the original distribution function *F*.

Theorem 1: **Extreme Value Theorem**

Let $M_n = \max\{\xi_1, \xi_2, \cdots \xi_n\}$, *where* $a_n > o, b_n$ ξ_i *are i.i.d. ramdom variables. If for some constants* $a_n > o, b_n$, *we have*

$$P\{a_n(M_n - b_n) \le x\} \to G(x)$$

for so nondegenerrate G, thhe G is one of the three extreme value types:

Type I: $G(x) = \exp(-e^{-x}), -\infty < x < \infty;$

Type II: $G(x) = \begin{cases} 0 & x \le 0, \\ \exp(-x^{-\alpha}), & x > 0, \end{cases}$ *for some* $\alpha > 0;$

Type III: $G(x) = \begin{cases} \exp(-(-x)^{\alpha}), & x \le 0, \\ 0 & x > 0, \end{cases}$ *for some* $\alpha > 0$.

Conversely, each distribution function G of extreme value type may appear as a limited in

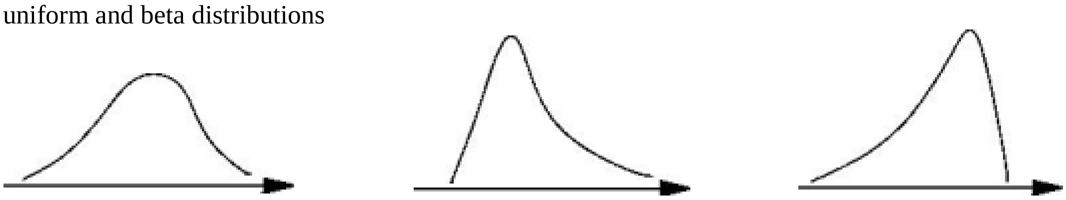
$$P\{a_n(M_n - b_n) \le x\} \to G(x)$$

and appears when G itself is the distribution of each ξ_i.

Jenkinson and von Mises suggested the following one-parameter representation

$$G(x) = \begin{cases} \exp\{(-(1+\xi x))^{-1/\xi}\} & \xi \ne 0; \\ \exp(-e^{-x}), & \xi = 0. \end{cases}$$

This generalization is known as extreme value distribution (*GEV*). In *GEV*, ξ is the shape parameter.
When ξ > 0 (Type II), it means the distribution has the heavy-tailed distributions whose tails decay like power functions such as the Pareto, Student's t, Cauchy, Burr, log-gamma, and Frchet distribution.
When ξ=0 (TypeI), it means the tails of distributions decay exponentially and we can call them thin-tailed distributions. Some good examples are normal, exponential, gamma and log-normal distributions.
When ξ<0 (TypeIII), the distributions are called short-taile ddistributions with finite right end point like the uniform and beta distributions



0 TypeI 0 Type II 0 Type III

1.2.2 The distribution of exceedances

Now we consider estimating the distribution function F_u of values of *x* above a certain threshold *u*.

The distribution function F_u is called the conditional function(*cedf*) and is defied as $F_u(y) = P(X - u \le y | X > u), 0 \le y \le x_F - u$, where *X* is a random variable, *u* is a given threshold, *y*=*x*-*u* are the excesses and $x_F \le \infty$ is the right endpoint of *F*. We verify that F_u can be written in terms of *F*, i.e.

$$F_u(y)=\frac{F(u+y)-F(u)}{1-F(u)}=\frac{F(x)-F(u)}{1-F(u)}.$$

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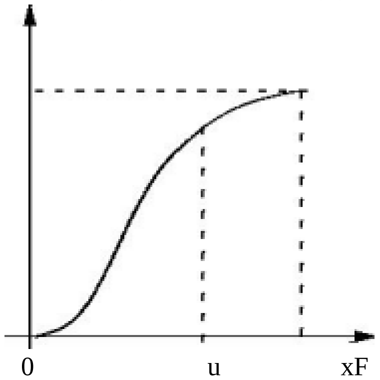

Figure 1

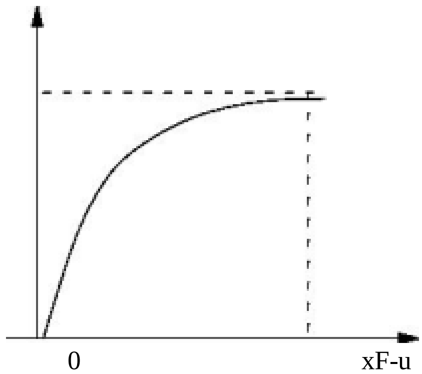

Figure 2

EVT provides us with a powerful result about the cedf which is stated in the following theorem:
Theorem 2 (Pickands(1975), Balkema and De Haan(1974)): *For a large class of underlying distribution functions F the conditional excess distribution function* $F_u(y)$*, for u large, is well approximated by*

$$F_u(y)\approx G_{\xi,\beta}(y), u\to\infty,$$

where

$$G_{\xi,\beta}(y)=\begin{cases}1-(1+\frac{\xi}{\beta}y)^{-1/\xi}, & \xi\neq 0\\ 1-e^{-y/\beta}, & \xi=0\end{cases}$$

For $y\in[0,x_F-u)]$ *if* $\xi\geq 0$*, and* $y\in[0,-\frac{\beta}{\xi}]$ *if* $\xi<0$. $G_{\xi,\beta}$ *is so-called generalized Pateto distribution (GPD).*

1.2.3 Risk measures without given distribution function

Given a random variable *X* (e.g. the uncertain loss of invested risk asset). We do not know the distribution of *X*. Following the extreme value theory, the limiting distribution of the random variable *X* above a threshold *u* is *GPD*. Now we will use above extreme value theory to derive analytical expressions for VaR_α and $CVaR_\alpha$.

First we write *F(x)* in terms of F_u, i.e.

$$F(x)=(1-F(u))F_u(y)+F(u).$$

Then replace F_u by the *GPD*, we have

$$F(x)=(1-F(u))(1-(1+\frac{\xi}{\beta}(x-u))^{-1/\xi})+F(u).$$

Given the confidence level $1-\alpha$,

$$VaR_\alpha=F^{-1}(1-\alpha)=u+\frac{\beta}{\xi}((\frac{\alpha}{1-F(u)})^{-\xi}-1).$$

Here, we can give the value of *F(u)* by the estimate $(n-N_u)/n$, where *n* is the number of observations and N_u the number of observations above the threshold *u*.

To value the condition value at risk, let us rewrite $CVaR_\alpha$ as:

$$CVaR_\alpha=E(X|X>VaR_\alpha)=VaR_\alpha+E(X-VaR_\alpha|X>VaR_\alpha),$$

where the second term on the right is the mean of the excess distribution $F_{VaR_\alpha}(y)$ over the threshold VaR_α.

It is known that the mean excess function for the *GPD* with parameter ξ < 1 is

$$e(z) = E(y - z | y > z) = \frac{\beta + \xi z}{1 - \xi}, \beta + \xi z > 0.$$

Therefore,

$$\begin{aligned} CVaR_\alpha &= VaR_\alpha + E(X - VaR_\alpha | X > VaR_\alpha) \\ &= VaR_\alpha + E(y - (VaR_\alpha - u) | y > VaR_\alpha - u) \\ &= VaR_\alpha + \frac{\beta + \xi(VaR_\alpha - u)}{1 - \xi} \\ &= \frac{VaR_\alpha}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi}. \end{aligned}$$

There is still a problem, that is how to select the threshold u and how to estimate the parameters ξ, β. One graphical tool called the sample mean excess plot is helpful for the selection of the threshold *u* (see Manfred Gilli, Evis Kellezi (2003)). Maximum likelihood estimation is used to estimate the parameters of *GPD*.

II. Portfolio Selection Models

We consider a market with *n* available securities. An investor with initial wealth W_0 seeks to improve his wealth status by investing his wealth into these *n* risky securities at the beginning of time *0*. The investment lasts for a period time *T*. Let $P_{i,T}$, $P_{i,o}$ be the price of the ith asset at time *T* and *0*. Also let R_i be the random rate of return of the ith security *(i = 1, . . . ,* n) and $R = (R_1, \cdots R_n)^T$ T the returns' vector. Therefore, $R_i = (P_{i,T} - P_{i,o}) / P_{i,o} \approx \ln P_{i,T} - \ln P_{i,o}$. The mean and covariance of the returns are assumed to be known, $\mu_i = E(R_i)$, and $\sigma_{i,j} = Cov(R_i, R_j), i, j = 1, \cdots n.$

Let x_i be the proportion of wealth the investor invests in the *i*th security, and $\sum_{i=1}^{n} x_i = 1$. Denote the decision vector in portfolio selection by $x = (x_1, \cdots x_n)^T$, mean of returns vector by $\mu = (\mu_1, \cdots \mu_n)^T$. Then, the random rate of return from holding securities is $\nu = \sum_{i=1}^{n} x_i R_i$. The mean and variance of ν are

$$E[\nu] = \mu^T x$$

And

$$VaR(\nu) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij} = x^T C x,$$

With

$$C = \begin{pmatrix} \sigma_{1,1} & \cdots & \sigma_{1,1} \\ \vdots & \ddots & \vdots \\ \sigma_{1,1} & \cdots & \sigma_{1,1} \end{pmatrix}.$$

We will introduce four different portfolio selection models as follows,

Model I: Mean-Variance Model can be formulated as

$$\begin{cases} \min_x & Var(\upsilon) \\ s.t. & E(\upsilon) \geq \varepsilon \end{cases}$$

where $\varepsilon > 0$ is a lower bound of the expected return of portfolio.

Model II: Safety-first Model can be formulated as

$$\min_x P(\upsilon \leq d)$$

where *d* is the upper disaster level. Following the Bienayme-Tchebycheff inequality, $P(\upsilon \leq d) = P(\upsilon - E(\upsilon) \leq d - E(\upsilon)) \leq \frac{Var(\upsilon)}{(E(\upsilon) - d)^2}$. Therefore, this optimization problem is equivalent to

$$\max_x \frac{(E(\upsilon)-d)^2}{Var(\upsilon)}.$$

Model III: Mean-VaR Model is introduced to allocate financial assets byminimizing *VaR* of the portfolio with the confidence level $1-\alpha$ and restrained the expected rate of return above a lower bound ε. The focus on *VaR* as the appropriate measure of portfolio risk allows investors to treat losses and gains asymmetrically.

$$\begin{cases} \min_x & VaR_\alpha(\upsilon) \\ s.t. & E(\upsilon) \ge \varepsilon \end{cases}.$$

Here $VaR_\alpha(\upsilon)$ is determined by the distribution of return of the portfolio $\upsilon = \sum_{i=1}^n x_i R_i$, i.e. $VaR_\alpha(\upsilon) = -F_{x,R}^{-1}(\alpha)$. We can verify that $F_{x,R}(\upsilon)$ is determined by the decision variable *x* and random variable *R*.

Model IV: Mean-*CVaR* Model is to minimize *CVaR* with a constraint of the expected return of portfolio.

$$\begin{cases} \min_x & CVaR_\alpha(\upsilon) \\ s.t. & E(\upsilon) \ge \varepsilon \end{cases}.$$

Similar to the VaR_α in the mean-*VaR* model, here $CVaR_\alpha$ is the conditional mean of loss of portfolio, i.e. $-\upsilon$, given that $-\upsilon$ is bigger than VaR_α, i.e.

$$CVaR_\alpha(\upsilon) = -E(\upsilon \mid \upsilon \le -VaR_\alpha).$$

2. Potential and limitation of using different risk measures in portfolio selection

2.1 Potential and limitation of Mean-Variance model

Mean-Variance model have some good properties which make it popular:

(i) It is consistent with the principle of ‘maximization of expected utility’ if the rate return of assets follows multi-variate normal distribution, which was usually considered to be a valid assumption for common stocks;

(ii) Quadratic programming problems, which is formulated from the singleperiod mean-variance model, can be simply solved by mathematical programming; The limitation of mean-variance model is that the assumption of normal distribution in this model is not suitable for some non-symmetric distributed instruments, such as option, bond, and some stocks. This limitation lead tothe mean-variance model underestimates the risk of portfolios.

2.2 Potential and limitation of Safety-first model

Safety-first model has some good properties compared with mean-variance model:

(i) It measures the downside risk, and put the risk of portfolio in a crucial rule, which is helpful to the investors who are principally concerned with avoiding a possible ‘disaster’;

(ii) Minimizing the chance of disaster in the safety-first model can be interpreted as maximizing expected utility if the utility function has only two values, i.e. one if disaster does not occur and zero if it does;

(iii) In the special case where d is equal to the gross return on riskless investment, safety-first model and the mean-variance model lead to the same results, i.e., the optimal unlevered portfolios constructed under each of the rules are identical (see Hain Levy and Marshall Sarmat (1972)).

2.3 Potential and limitation of *VaR*

VaR (value-at-risk), a relatively new lower partial risk measure, is widely used for the measurement of market risk. The following are its good properties:

(i) It is regardless of the distribution of the underlying assets, so that it is more consistent with the ‘maximization of expected utility principle’ than mean-variance model;

(ii) When assume that the distribution of the underlying assets are normal distribution, the mean-*VaR* model has the same result with the meanvariance model, i.e., a mean-*VaR* efficient portfolio is mean-variance efficient (see Jin Wang (2000)).

There are three shortcomings,

(i) *VaR* measures are lack of coherence, e.g. it lacks the sub-additivity property. So it has difficulties to aggregate individual risks, and sometimes discourage diversification (see Artzner et al (1998));

(ii) It is only focusing on controlling the probability of loss, rather than its magnitude. Hence, the expected losses, conditional on the states where there are large losses, may be higher sometimes;

(iii) the *VaR* of a portfolio x, VaR_x is not a convex function of *x*, so that it is very difficult to minimize VaR_x

over $x \in X$. We thus rely on a genetic algorithm or a heuristic approach (see Larsen, Mausser and Uryasev (2001)).

2.4 Potential and limitation of *CvaR*

Compare with *VaR*, *CVaR* has nice properties both theoretically and computationally as following:
(i)As same as *VaR*, it is regardless of the distribution of the underlying assets, so that it is more consistent with the 'maximization of expected utility principle' than mean-variance model;
(ii) *CVaR* is a coherent measure of risk, and minimizing $CVaR_x$ can be converted to minimizing a convex function (we will introduce the methodology latter). Therefore, we can use convex minimization algorithms to solve the mean-*CVaR* model.

2.5 Potential and limitation of *EVT*

EVT (extreme value theory) is currently in the focus of interest in quantitative risk management. It can be used to analysis rare events and value *VaR*, *CVaR* without the assumption of original distribution function. It gives a methodological toolkit for issues like skewness, fat tails, rare events, etc, and makes the best use of whatever data we have about extreme phenomena. At the same time, its limitations are pointed out as belows:
(i) In order to estimate way in the tails, one has to verify the tail model, which is very difficult;
(ii) For a *EVT-VaR* estimation, one has to set appropriate thresholdabove which the data are to be used for tail estimation. There is no canonical choice;
(iii) Handling extremes for high dimensional portfolios is difficult.

III. Methodology

The most difficult part of portfolio selection is to find a useful optimization method to solve it, get the optimal investment policy and derive the efficient frontier.

With knowing the distribution function, the mean-variance model and safety-first model can be handled by quadratic programming. To handle the mean-*VaR* model, there are three methods. Mostly, approaches to calculating *VaR* rely on linear approximation of the portfolio risks and assume a joint normal distribution of the underlying market parameters. Also, historical or Monte Carlo simulation-based tools are used when the portfolio contains nonlinear instruments such as options. As to mean-*CVaR* model, R.tyrrell Rockfellar, Stanislav Uryasev (1999) provide an approach to convert the original problem to a linear convex programming. The first part of this section will introduce this approach.
Without the assumption of the distribution of underlying assets, we use *EVT* to measure these risks, so as to handle the portfolio selection models. This approach will be introduced in the second part of this section.

3.1 CVaR minimization approach

Let f(x, y) be a loss function (i.e. the loss of the portfolio) depending upon a decision vector $x = (x_1, \cdots, x_n)$ (i.e. the amount of wealth invested in assets 1,… , n) and a random vector $y = (y_1, \cdots, y_m)$ (i.e. the uncertain returns ofassets).
For each x, we denote by $\Psi(x,\cdot)$ on $\Re$ the resulting distribution function for the loss *f(x, y)*, i.e.,

$$\Psi(x,\zeta) = P\{y | f(x,y) \ge \zeta\}.$$

The α-*VaR* of the loss associated with a decision *x* is the value $\zeta_\alpha(x) = \min\{\zeta | \Psi(x,\zeta) \ge \alpha\}$.

The α-*CVaR* of the loss associated with a decision *x* is the value $\phi_\alpha(x)$ =mean of the α-tail distribution of *f(x, y)*, where the distribution in question is the one with distribution function $\Psi(x,\cdot)$ defined by

$$\Psi(x,\zeta) = \begin{cases} 0, & \zeta < \zeta_\alpha(x); \\ \dfrac{\psi(x,\zeta) - \alpha}{1-\alpha}, & \zeta \ge \zeta_\alpha(x). \end{cases}$$

We assume that the random vector *y* has a probability density function *p(y)*. Therefore, the resulting distribution function of the loss *f(x, y)* is $\Psi(x,\zeta) = \int_{f(x,y)\le\alpha} p(y)dy$. Also we assume that the probability distributions are such that no jumps occur, or in other words, that $\Psi(x,\zeta)$ is everywhere continuous with respect to α.
Following the definition of V aR and *CVaR*, we know that the *VaR* function $\zeta_\alpha(x)$, which is the percentile of the loss distribution with confidence level α, is the smallest number such that $\Psi(x, \zeta_\alpha(x)) = \alpha$. *CVaR*,

denoted by $\phi_\alpha(x)$, can be expressed by

$$\phi_\alpha(x) = (1-\alpha)^{-1}\int_{f(x,y)>\zeta_\alpha(x)} f(x,y)p(y)dy.$$

The key to the approach is a characterization of $\phi_\alpha(x)$ and $\zeta_\alpha(x)$ in terms of the function F_α, which is defined by

$$F_\alpha(x,\zeta) = \zeta + (1-\alpha)^{-1}\int_{y\in\Re^m}[f(x,y)-\zeta]^+ p(y)dy.$$

Theorem 3: *As a function of* ζ, $F_\alpha(x,\zeta)$ *is convex and continuously differentiable. The α-CVaR of the loss associated with any x ∈ X can be determined from the formula*

$$\phi_\alpha(x) = \min_{\zeta\in\Re} F_\alpha(x,\zeta).$$

In this formula the set consisting of the values of ζ *for which the minimum is attained, namely*

$$A_\alpha(x) = \arg\min_{\zeta\in\Re} F_\alpha(x,\zeta).,$$

is a nonempty, closed, bounded interval (perhaps reducing to a single point), and the α-VaR of the loss is given by $\zeta_\alpha(x)$ *=left end point of* $A_\alpha(x)$. *In particular, one always has*

$$\zeta_\alpha(x) = \arg\min_{\zeta\in\Re} F_\alpha(x,\zeta) \text{ and } \phi_\alpha(x) = F_\alpha(x,\zeta_\alpha(x)).$$

Theorem 4: *Minimizing the α-CVaR of the loss associated with x over all x ∈ X is equivalent to minimizing* $F_\alpha(x,\zeta)$ *over all* $(x,\zeta)\in X\times\Re$, *in the sense that*

$$\min_{x\in X}\phi_\alpha(x) = \min_{(x,\zeta)\in X\times\Re} F_\alpha(x,\zeta).$$

where moreover a pair (x^*,ζ^*) *achieves the second minimum if and only if* $x*$ *achieves the first minimum and* $\zeta^*\in A_\alpha(x^*)$. *In particular, therefore, in circumstances where the interval* $A_\alpha(x^*)$ *reduces to a single point, the minimization of* $F\ (x,\zeta)$ *over* $(x,\zeta)\in X\times\Re$ *produces a pair* (x^*,ζ^*), *not necessarily unique, such that* $x*$ *minimizes the α-CVaR and* $(\zeta^*$ *gives the corresponding α-VaR.*

Furthermore, $F_\alpha(x,\zeta)$ is convex with respect to (x,ζ), and $\phi_\alpha(x)$ is convex with respect to *x*, when *f(x, y)* is convex with respect to *x*, in which case, if the constraints are such that *X* is a convex set, the joint minimization is an instance of convex programming.

Let us consider the case in which an analytical representation of the density function *p(y)* is not available, but we have *J* scenarios, $y_j, j=1,\cdots J$, sampled from the density *p(y)*. In this case, the function $F_\alpha(x,\zeta)$ can be calculated approximately as follows

$$\tilde{F}_\alpha(x,\zeta) = \zeta + ((1-\alpha)J)^{-1}\sum_{j+1}^{J}(f(x,y_j-\zeta)^+.$$

If the feasible set *X* is convex, the optimization problem with the *CVaR* performance function can be solved using non-smooth optimization techniques. Moreover, if the function *f(x, y)* is linear to x, these problems can be solved using LP techniques. By replacing the terms $(f(x,y_j-\zeta)^+$ by auxiliary variables z_j, and imposing constraints $z_j \geq f(x,y_j-\zeta, z_j \geq 0$, we can reduce optimization of the portfolio problem to the following LP problem and solve it by CPLEX:

$$\begin{cases} \min\limits_{x\in X,\zeta\in\Re,z\in\Re^J} & \zeta + ((1-\alpha)J)^{-1}\sum_{j-1}^{J} z_j \\ s.t. & J^{-1}\sum_{j-1}^{J} f(x,y_j) \leq V \\ & x\in X \\ & z_j \geq f(x,y_j-\zeta, z_j \geq 0 \end{cases}.$$

5.2 *EVT* approach

First, we consider the Safety-first model min $P(\upsilon\leq d)$. Given $P(\upsilon\leq d) = F_\upsilon(d) = 1-\tilde{F}(-d)$, where *F* is the cumulative distribution function of gain

υ and $\tilde{F}$ is the cumulative distribution function of loss $-\upsilon$. Following *EVT*, $\tilde{F}(x) \to G_{\xi}(\frac{x-\mu}{\sigma})$, where μ and σ are normalized parameters. Given *m* observations of $R_i, \{r_i^j\}_{j=1,\cdots m}$, we can solve the model in the following two steps.

Step 1. $\max_{\xi,\mu,\sigma} L(\xi,\mu,\sigma;\upsilon) = \prod_j \log(g(-\upsilon_j))$, where

$$g(\xi,\mu,\sigma;\upsilon) = \frac{1}{\sigma}(1+\xi\frac{\upsilon-\mu}{\sigma})^{-1/\xi-1}\exp(-(1+\xi\frac{\upsilon-\mu}{\sigma})^{-1/\xi}) \text{ and } \upsilon_j = \sum_{i=1}^n x_i r_i^j \;;$$

Step 2. $\max_x G(\xi^*(x),\mu^*(x),\sigma^*(x);-d)$ or $\max_x \exp(-(1+\xi\frac{\upsilon-\mu^*(x)}{\sigma^*(x)})^{-1/\xi^*(x)})$.

Then we consider Mean-*VaR* model and Mean-*CVaR* model. As we know that $VaR_x(\upsilon) = -F_{\upsilon}^{-1}(\alpha) = \tilde{F}^{-1}(1-\alpha)$, where *F* is the cumulative distribution function of gain υ and $\tilde{F}$ is the cumulative distribution function of loss $-\upsilon$. Given a preselected threshold *u*, $\tilde{F}_u$ is the conditional excess distribution function of $\tilde{F}$. Following *EVT*, it has a limited distribution *GPD*, $\tilde{F}_u(y) \to H_{\xi,\sigma}(y)$.

Suppose $\{-r_i^j\}_{j=1,\cdots,m}$ are *m* observations of $-R_i$ above the threshold *u*. W can handle the above two models in two steps.

Step 1 $\max_{\xi,\beta} L(\xi,\beta;y)$ where

$$L(\xi,\beta;y) = \begin{cases} -n\log\beta - (\frac{1}{\xi}+1)\sum_{j=1}^m \log(1+\frac{\xi}{\beta}y_j), & \xi \neq 0 \\ -n\log\beta - \frac{1}{\xi}\sum_{j=1}^m y_j, & \xi = 0, \end{cases}$$

And $y_j = -\sum_{i=1}^n x_i r_i^j$;

Step 2 $\min_x VaR(\xi^*(x),\beta^*(x);\alpha) = \min_x u + \frac{\beta^*(x)}{\xi^*(x)}((\frac{\alpha}{1-F(u)})^{-\xi^*(x)} - 1);$ or

$$\min_x CVaR(\xi^*(x),\beta^*(x);\alpha) = \min_x u + [\frac{\beta^*(x)}{\xi^*(x)}((\frac{\alpha}{1-F(u)})^{-\xi^*(x)} - 1) + \beta^*(x)]\frac{1}{1-\xi^*(x)}.$$

IV. Conclusion

EVT techniques make it possible to concentrate on the behavior of the extreme observations. It does not assume a particular model for returns but lets the data speak for themselves to fit the distribution tails. However, until now, only approximation procedures with *EVT* method has been used to allocating assets because of the difficulty in forming the optimization problem (see Younes Bensalah 2002)). We have built a set of complex optimization problems for portfolio selection with *EVT*. Solving these problems may be a contributed work, and it may be more interesting to extend this problem to a multi-period case.

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