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Marshall-Olkin Weibull Fréchet Max-Min Processes

Catherine Thomas[1;] and Seetha Lekshmi V.[2;†]
[1]*Dept. of Statistics, Govt. College, Kottayam, Kerala, India*
[2]*Dept. of Statistics, Nirmala College, Muvattupuzha, Kerala, India*

Abstract: *In this paper we consider Weibull Fréchet distribution and study its properties. The Marshall-Olkin Weibull Fréchet distribution is developed and studied in detail. AR(1) models with Weibull Fréchet marginal distribution are introduced. As a further extension general theory of Max-Min AR(1) processes are also developed and generalized it to Kth order. Max-Min process with Weibull Fréchet marginal distribution is introduced and studied. Applications are also discussed.*

Key words: *AR(1) models, Max-Min processes, Marshall-Olkin Weibull Fréchet distribution, Weibull Fréchet distribution.*

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I. Introduction

Weibull-Fréchet distribution was introduced recently by Afify et al. (2016). It has applications in engineering, medicine, and other areas of research.Both Weibull and Fréchet distributions play a significant role in extreme value theory and can be used to model accelerated life times, earth quakes, floods, rainfalls, wave heights, usual speeds etc.

Marshall and Olkin(1997) introduced a generalized family of distributions and applied the results to extend exponential and Weibull distributions. Many researchers have recently studied Marshall-Olkin family of distributions and applied in various contexts such as reliability analysis, time series modeling etc. For details see Jayakumar and Thomas (2008), Sankaran and Jayakumar (2006), Krishna et al.(2013a,b), Jose et al.(2010,2011,2014).These distributions offer wide flexibility and can be used to model data from various areas.

Autoregressive processes with non-Gaussian marginal distributions have received much attention in recent years. Lewis and McKenzie(1991) introduced minification processes and their general theory. Alice and Jose(2004), Seethalekshmi and Jose(2004,2006), Jose and Naik(2010), Jose and Remya(2015) etc are some recent works in this respect.

This paper is organized as follows. In section 2,Weibull fréchet distribution is reviewed. In section 3, we introduce the Marshall-Olkin Weibull Fréchet distribution and studied its important properties. AR(1) models with Weibull Fréchet marginal distribution are introduced in section 4. As a further extension, general theory of Max-Min AR(1) processes are also developed in section 5 and generalized it to the kth order. In section 6, Max-Min process with Weibull Fréchet marginal distribution is introduced and studied. Applications are discussed in section 7.

II. Weibull Fréchet Distribution

The cdf of four parameter WFr (Weibull Fréchet) distribution is given by

$$F(x) = 1 - exp(-a(exp[(\alpha/x)^{\beta} - 1])^{-b}) \tag{1}$$

It is a generalization of many distributions like exponential Fréchet Weibull-inverse exponential Weibull-inverse Rayleigh etc.

The pdf is

$$f(x) = ab\beta\alpha^{\beta}exp[-b(\alpha/x)^{\beta}][1 - exp-(\alpha/x)^{\beta}]^{-b-1}exp(-a(exp(\alpha/x)^{\beta} - 1)^{-b}) \tag{2}$$

The HRF is given by

$$h(t) = \frac{f(t)}{\bar{F}(t)} = ab\beta\alpha^{\beta}exp[-b(\alpha/t)^{\beta}][1 - exp[-(\alpha/t)^{\beta}]]^{-b-1} \tag{3}$$

where α is a scale parameter representing the characteristic life and β, a and b are shape parameters representing the different patterns of the WFr distribution.
The CHR(Cumulative Hazard Rate function) of X is

$$H(x) = a(exp[(\alpha/x)^{\beta}] - 1)^{-b} \tag{4}$$

Now we introduce a generalization called Marshall-Olkin WFr distribution.

3. Marshall- Olkin WFr (MOWFr) Distribution and its Properties

In this section a new probability model known as Marshall-Olkin WFr (MOWFr) distribution is devloped. Various properties of the distribution and hazard rate functions are considered. The corresponding time series models are developed to illustrate its application in time series modeling.

Let $\bar{F}(x)$ be the survival function of a given distribution. The survival function of the Marshall-Olkin distribution obtained by introducing a new parameter p is given by

$$\bar{G}(x) = \frac{p\bar{F}(x)}{1-(1-p)\bar{F}(x)}, \ -\infty < x < \infty, \ \ 0 < p < \infty.$$

Clearly when $p = 1$, we get the standard form of the survival function. Corresponding pdf is given by

$$g(x) = \frac{p\ f(x)}{[1-(1-p)\bar{F}(x)]^2}, \ -\infty < x < \infty, \ \ 0 < p < \infty$$

and the hazard rate function is given by

$$h(t) = \frac{h_F(t)}{1-(1-p)\bar{F}(t)}, \ -\infty < t < \infty, \ \ 0 < p < \infty$$

where $h_F(t)$ denote the hazard rate function of the original model with distribution function F. The survival function of the Marshall-Olkin distribution obtained by introducing a new parameter p is given by

$$\bar{G}(x) = \frac{pexp(-a(exp[(\alpha/x)^{\beta}]-1)^{-}b)}{1-(1-p)exp(-a(exp[(\alpha/x)^{\beta}]-1)^{-b})}, \ -\infty < x < \infty, \ \ 0 < p < \infty.$$

Clearly when $p = 1$, we get the standard form of the survival function. Corresponding pdf is given by

$$g(x) = \frac{p\ ab\beta\alpha^{\beta}exp[-b(\alpha/x)^{\beta}][1-exp[-(\alpha/x)^{\beta}]]^{-b-1}exp(-a(exp[(\alpha/x)^{\beta}]-1)^{-b})}{[1-(1-p)exp(-a(exp[(\alpha/x)^{\beta}]-1)^{-b})]^2}$$

,

$$-\infty < x < \infty, \ \ 0 < p < \infty$$

and the hazard rate function (HRF) is given by

$$h(x) = \frac{ab\beta\alpha^{\beta}exp[-b(\alpha/x)^{\beta}][1-exp[-(\alpha/x)^{\beta}]]^{-b-1}}{1-(1-p)exp(-a(exp[(\alpha/x)^{\beta}-1))^{-b})}, \ -\infty < x < \infty, \ \ 0 < p < \infty$$

Figure 1 shows the pdf and HRF ofMOWFr distribution.

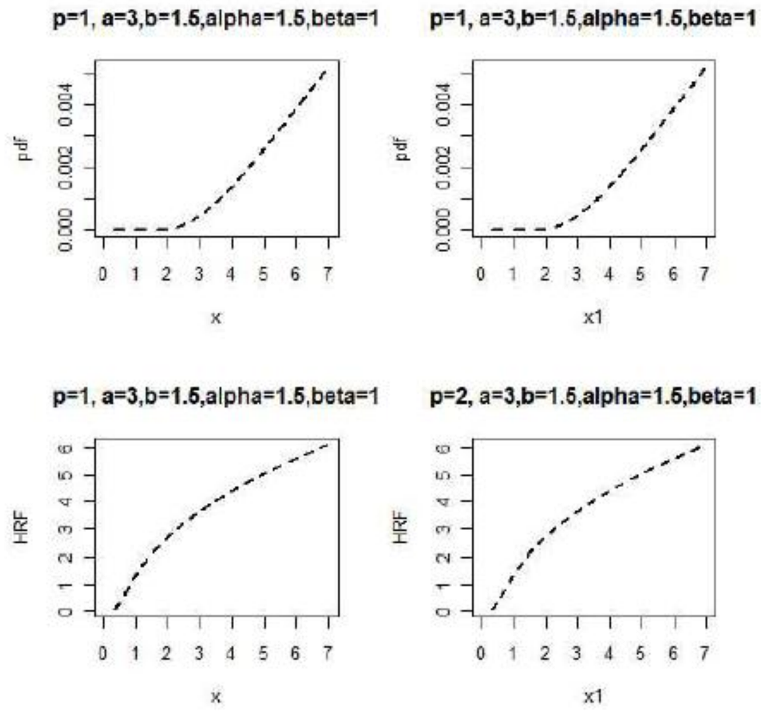

Figure 1

4. AR(1) models with MOWFr distribution

Two stationary Markov processes with similar structural forms which is useful in hydrological applications was introduced by Tavares(1977,1980). The various aspects on first order autoregressive minification was discussed in Lewis and Mc Kenzie(1991). In this section we develop autoregressive minification processes of order one and order k with minification structures are discussed with MOWFr distribution as stationary marginal distribution. We call the process as MOWFr AR(1) process. Now we have the following theorem.

Theorem 4.1 *Consider an $AR(1)$ structure given by*

$$X_n = \begin{cases} \varepsilon_n, & w.p.\ p_1 \\ \min(X_{n-1}, \varepsilon_n), & w.p.\ 1-p_1 \end{cases}$$

where w.p. denotes 'with probability', $0 < p_1 < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then $\{X_n\}$ is a stationary Markovian $AR(1)$ process with MOWFr marginal if and only if $\{\varepsilon_n\}$ is distributed as WFr distribution.

Proof: From the given structure it follows that

$$\bar{F}_{X_n}(x) = p_1 \bar{F}_{\varepsilon_n}(x) + (1-p_1)\bar{F}_{X_{n-1}}(x)\ \bar{F}_{\varepsilon_n}(x).$$

Under stationary equilibrium, it reduces to

$$\bar{F}_X(x) = \frac{p_1 \bar{F}_\varepsilon(x)}{1-(1-p_1)\bar{F}_\varepsilon(x)}. \tag{5}$$

On substituting the survival function of error ε, for $0 < q < 1$, we get

$$\bar{F}_X(x) = \frac{p_1\, exp(-a(exp[(\alpha/x)^{\beta} - 1])^{-b})}{1 - (1 - p_1) exp(-a(exp[(\alpha/x)^{\beta} - 1])^{-b})}.$$

which resembles the survival function $\bar{G}_1(\cdot)$ of the MOWFr distribution. Conversely, if we take the survival function of the above form, we get the corresponding survival function of ε as

$$\bar{F}_{\varepsilon}(x) = exp(-a(exp[(\alpha/x)^{\beta} - 1])^{-b}) \tag{6}$$

which is WFr distribution under stationary equilibrium.

Theorem 4.2 *Consider an* $AR(1)$ *structure given by*

$$X_n = \begin{cases} X_{n-1}, & w.p.\ p_2 \\ \varepsilon_n, & w.p.\ p_1(1-p_2) \\ \min(X_{n-1}, \varepsilon_n), & w.p.\ (1-p_1)(1-p_2) \end{cases}$$

where $\{\varepsilon_n\}$ *is a sequence of i.i.d. random variables independently distributed of* X_n*. Then* $\{X_n\}$ *is a stationary Markovian* $AR(1)$ *process with MOWFr marginal if and only if* $\{\varepsilon_n\}$ *is distributed as WFr distribution.*

Proof: From the given structure it follows that

$$\bar{F}_{X_n}(x) = p_2 \bar{F}_{X_{n-1}}(x) + p_1(1-p_2)\bar{F}_{\varepsilon_n}(x) + (1-p_1)(1-p_2)\bar{F}_{X_{n-1}}(x)\, \bar{F}_{\varepsilon_n}(x).$$

On simplification we get, the same expression as in equation (6) under stationarity. Then the result is obvious.
The following theorem generalizes the results to a k^{th} order autoregressive structure.

Theorem 4.3 *Consider an* $AR(k)$ *structure given by*

$$X_n = \begin{cases} \varepsilon_n, & w.p.\ p_0 \\ \min(X_{n-1}, \varepsilon_n), & w.p.\ p_1 \\ \min(X_{n-2}, \varepsilon_n), & w.p.\ p_2 \\ \quad\vdots & \quad\vdots \\ \min(X_{n-k}, \varepsilon_n), & w.p.\ p_k \end{cases}$$

where $\{\varepsilon_n\}$ *is a sequence of i.i.d. random variables independently distributed of* X_n*,* $0 < p_i < 1$*,* $p_1 + p_2 + \cdots + p_k = 1 - p_0$*. Then the stationary marginal distribution of* $\{X_n\}$ *is MOWFr if and only if* $\{\varepsilon_n\}$ *is distributed as WFr distribution.*

Proof: From the given structure the survival function is given as follows:

$$\bar{F}_{X_n}(x) = p_0\, \bar{F}_{\varepsilon_n}(x) + p_1\, \bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x) + \cdots + p_k\, \bar{F}_{X_{n-k}}(x)\bar{F}_{\varepsilon_n}(x).$$

Under stationary equilibrium, this yields

$$\bar{F}_X(x) = p_0\, \bar{F}_{\varepsilon}(x) + p_1\, \bar{F}_X(x)\bar{F}_{\varepsilon}(x) + \cdots + p_k\, \bar{F}_X(x)\bar{F}_{\varepsilon}(x).$$

This reduces to

$$\bar{F}_X(x) = \frac{p_0\,\bar{F}_\varepsilon(x)}{1-(1-p_0)\bar{F}_\varepsilon(x)}.$$

Then the theorem easily follows by similar arguments as in Theorem 4.2.

5. The max-min AR(1) processes

We introduce a new model called the max-min process which incorporates both maximum and minimum values of the process. This has wide applications in atmospheric and oceanographic studies. The structure is given in the following theorem.

Theorem 5.1 *Consider an $AR(1)$ structure given by*

$$X_n = \begin{cases} \max(X_{n-1}, \varepsilon_n), & w.p.\ p_1 \\ \min(X_{n-1}, \varepsilon_n), & w.p.\ p_2 \\ X_{n-1}, & w.p.\ 1-p_1-p_2 \end{cases}$$

subject to the conditions $0 < p_1, p_2 < 1, p_2 < p_1$ and $p_1 + p_2 < 1$, where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then $\{X_n\}$ is a stationary Markovian $AR(1)$ max-min process with stationary marginal distribution $\bar{F}_X(x)$ if and only if $\{\varepsilon_n\}$ follows Marshall-Olkin distribution.

Proof: From the given structure it follows that

$$\begin{aligned} \bar{F}_{X_n}(x) = P(X_n > x) &= p_1\,P(\max(X_{n-1}, \varepsilon_n) > x) + p_2\,P(\min(X_{n-1}, \varepsilon_n) > x) \\ &\quad + (1-p_1-p_2)\,P(X_{n-1} > x) \\ &= p_1\left[1-(1-\bar{F}_{X_{n-1}}(x))(1-\bar{F}_{\varepsilon_n}(x))\right] + p_2\,\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x) \\ &\quad + (1-p_1-p_2)\bar{F}_{X_{n-1}}(x). \end{aligned}$$

Under stationary equilibrium, we get

$$\bar{F}_\varepsilon(x) = \frac{p_2\,\bar{F}_X(x)}{p_1 + (p_2-p_1)\bar{F}_X(x)} = \frac{p'\bar{F}_X(x)}{1-(1-p')\bar{F}_X(x)} \tag{7}$$

where $p' = \frac{p_2}{p_1}$. This has the same functional form of Marshall-Olkin survival function. The converse can be proved by mathematical induction, assuming that $\bar{F}_{X_{n-1}}(x) = \bar{F}_X(x)$.

6. The max-min process with WFr marginal distribution

To obtain the WFr max-min process, applying WFr survival function in equation (7), we obtain

$$\bar{F}_\varepsilon(x) = \frac{p'exp(-a(exp[(\alpha/x)^\beta - 1])^{-b})}{1-(1-p')exp(-a(exp[(\alpha/x)^\beta - 1])^{-b})},$$

which is the MOWFr survival function of distributions with $p' = \frac{p_2}{p_1}, p_2 < p_1$ and $p_1 + p_2 < 1$.

Now consider a more general autoregressive structure which includes maximum, minimum as well as the innovations and the process values.

Theorem 6.1 *Consider an $AR(1)$ structure given by*

$$X_n = \begin{cases} \max(X_{n-1}, \varepsilon_n), & w.p.\ p_1 \\ \min(X_{n-1}, \varepsilon_n), & w.p.\ p_2 \\ \varepsilon_n, & w.p.\ p_3 \\ X_{n-1}, & w.p.\ p_4 \end{cases}$$

with the condition that $0 < p_1, p_2, p_3, p_4 < 1, p_2 < p_1$ and $p_4 = 1 - p_1 - p_2 - p_3$, where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then $\{X_n\}$ is a stationary Markovian $AR(1)$ max-min process with stationary marginal distribution $\bar{F}_X(x)$ if and only if $\{\varepsilon_n\}$ follows Marshall-Olkin distribution.

Proof: From the given structure it follows that

$$\begin{aligned} P(X_n > x) &= p_1\, P(\max(X_{n-1}, \varepsilon_n) > x) + p_2\, P(\min(X_{n-1}, \varepsilon_n) > x) \\ &\quad + p_3\, P(\varepsilon_n > x) + (1 - p_1 - p_2 - p_3)\, P(X_{n-1} > x). \end{aligned}$$

This simplifies to

$$\begin{aligned} \bar{F}_{X_n}(x) &= p_1 \left[1 - (1 - \bar{F}_{X_{n-1}}(x))(1 - \bar{F}_{\varepsilon_n}(x)) \right] + p_2\, \bar{F}_{X_{n-1}}(x) \bar{F}_{\varepsilon_n}(x) \\ &\quad + p_3 \bar{F}_{\varepsilon_n}(x) + (1 - p_1 - p_2 - p_3) \bar{F}_{X_{n-1}}(x). \end{aligned}$$

Under stationary equilibrium, this reduces to

$$\bar{F}_\varepsilon(x) = \frac{(p_2 + p_3)\, \bar{F}_X(x)}{p_1 + p_3 + (p_2 - p_1)\bar{F}_X(x)} = \frac{\beta \bar{F}_X(x)}{1 - (1 - \beta)\bar{F}_X(x)} \tag{8}$$

where $\beta = \frac{p_2 + p_3}{p_1 + p_3}$. This has the same functional form of the Marshall-Olkin survival function. Conversely $\bar{F}_X(x)$ follows survival function of WFr distribution.

Remark: By substituting the survival function of WFr distribution in (8) we obtain the survival function of MOWFr distribution, as

$$\bar{F}_\varepsilon(x) = \frac{\beta exp(-a(exp[(\alpha/x)^\beta - 1])^{-b})}{1 - (1 - \beta) exp(-a(exp[(\alpha/x)^\beta - 1])^{-b})} \tag{9}$$

where $\beta = \frac{p_2 + p_3}{p_1 + p_3}$.

7. Applications

The Marshall-Olkin WFr (MOWFr) distribution studied in this paper can be used for modeling data from various areas such as statistical mechanics, financial contexts, communications engineering, entropy studies etc. The max-min auto regressive processes can be used for modeling time series data from hydro logical, financial and reliability contexts. They accommodate four components with respect to innovations, processes, minimum as well as maximum of the process values and offers wide flexibility in modeling real data sets.

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