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Variational Method Of Walther Ritz, Leonid Vitalievich Kantorovich And Functional Method Of Boris Grigorievich Galiorkin To Solve The Heat Equation

Marco Antonio Gutiérrez Villegas[1], Esiquio Martín Gutiérrez Armenta[2], Israel Isaac Gutiérrez Villegas[3 y 4], Alfonso Jorge Quevedo Martinez[5], Javier Norberto Gutiérrez Villegas[6], Francisco Javier Hernández Baraja[7] y Juan Manuel Figueroa Flores[8].

[1,2,5]*(Departamento De Sistemas, Área De Sistemas Computacionales, Universidad Autónoma Metropolitana Unidad Azcapotzalco, Cdmx, México).*
[3,6,7]*(División De Ingeniería En Sistemas Computacionales, Tese- Tecnm, México).*
[5]*(Departamento De Administración, Área De Matemáticas Y Sistemas, Universidad Autónoma Metropolitana Unidad Azcapotzalco, México).*
[4,8] *(Departamento De Ingeniería Y Ciencias Sociales, ESFM - IPN, CDMX, México).*

Abstract:
Piere Simon Laplace is an efficient method as long as there is enough information about its initial conditions, but when different boundary conditions are used, its application of these techniques becomes complex to find the inverse transform which directly gives the result, for this type of problems, simpler methods should be applied, which are not complicated like the mentioned techniques, so the variational methods Walther Ritz, Leonid VitálievichKantoróvich and the functional formulation of BorísGrigórievichGaliorkinare used, these are approximate analytical solutions that can be applied to all the mechanics of a continuous medium, a particular case will be used to determine the temperature distribution in an electric heater with uniform internal generation.

Background*:The Fourier heat equation and the birth of the series, Jean Baptiste Joseph Fourier is often said to have given rise to climatological science. His 1827 article "M' emorie modern sur les Températures du Globe Terrezte et des espacesPlanétaries" (I translated it into English as "on terrestrial temperatures and interplanetary spaces"), [1] contains in his work a complete study from Aristotle, against an approximation to the irrational number e, Natural logarithms, which at that time were known as "Hyperbolic logarithms". An application for the number e that was independent of logarithms was first formulated by Jakob Bernoulli (1655-1705) in the context of a compound interest problem in the late 17th century. See for more details [2,3] uses the Fourier expansion for the heat equation like "The infinite Fourier series, proof of the irrationality of e", but this methodology is used to calculate in Fourier series a function with finite continuities Applying them to solve heat transfer problems in one, two and three dimensions, as well as time, with different boundary conditions, this makes it more difficult to solve algebraically.*

Materials and Methods*: The Piere Simon Laplace method is used when the initial conditions are available, however when the problem has different boundary conditions its application becomes complex, so variational methods such as Walther Ritz, Leonid VitálievichKantoróvich and the formulation are used. functional by BorísGrigórievichGaliorkin.*

Results*: The solution by the Kantorovich Method is better than the Ritz and Galerkin Method as shown in the table because of the percentage errors. The Ritz and Galerkin Method are simpler to calculate, since one only has to integrate and derive with respect to the unknown constants. These give the same results and error because the same test function was used. The Kantorovich method, you have to integrate, then use the Euler-Lagrange equation, resulting in a differential equation, in this case ordinary with constant coefficients, which has to be solved by the Method of undetermined coefficients or variation of parameters, which makes it more complicated to solve the problem. In the Kantorovich Method, it is necessary to know the behavior of the phenomenon (in this case of the problem). In order to propose a solution in one of the x or y directions, while in the other cases this is not necessary.*

Conclusion:*The Kantorvich method is a simple method, but the mathematical procedure is complicated, compared to the other two, so it can be concluded that the other two can be used with more terms to improve the approximation, because its development is simple to obtain a better approximation.*

Key Word*:BorísGrigórievichGaliorkinMethod, Walther Ritz Method, VitálievichKantoróvichMethod,*

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I. Introduction

In 1870 Lord Rayleigh's classic, which was based on the approximation of a single function, this was extended by Ritz in a work published in 1908 in the year 1894, Rayleigh had the idea of optimizing the eigenvalues by minimizing with respect to an exponential parameterization, The Rayleigh-Ritz and Galerkin methods are extremely powerful techniques that allow obtaining an approximate analytical solution to a variety of problems in physics, continuous medium mechanics, medicine, biology, control, chemical processes, etc. On the other hand, the optimization of these methods is detailed; these functions that are used in their development must meet the fact that they are orthogonal, as well as meet the boundary conditions. See full article by [4]. Variational methods such as Galerkin's are only applied directly to the differential equation and are taken to integral form by proposing orthogonal polynomials that satisfy the boundary conditions [5] that are based on the minimization of a functional associated with the given problem, which It is nothing other than the total energy of the physical system. The finite method [6] can be applied to this. It is good for when you have to work with problems in a permanent state, but in the transient one if the number parameter is passed. Fourier begins to do things that do not agree with the solution.

II. Application

You have the problem of an electric heater. With the following information. Let be a rectangular cross section $(-a,a)X(-b,b)$ with a uniform (constant) internal heat source $\dot{q}=\frac{w}{m^3}$(Heat generated per unit volume) whose boundaries maintained at the temperature zero, the thermal conductivity $\lambda=\frac{W}{m k^o}.\frac{Watt}{metrogradosKelvin}$ is what represents the ability of the substance or material to conduct heat. Table no 1 shows some elements of thermal conductivity.

Table no 1: Some values of thermal conductivities.

Metals, at 25 ºC		Gases, at 25 ºC		Other materials	
Substance	k (W/mK)	Substance	k (W/mK)	Substance	k (W/mK)
Aluminum	238	Air	0.0234	Asbestos	0.08
Copper	397	Helium	0.138	Concrete	0.8
Gold	314	Hydrogen	0.172	Diamond	2300
Iron	79.5	Nitrogen	0.0234	Glass	0.84
Lead	34.5	Oxygen	0.0238	Rubber	0.2
Silver	427			Wood	0.08 to 0.16
Brass	110			Cork	0.42
				Human Tissue	0.2
				Water	0.56
				Ice	2

For this particular case, the temperature distribution will be determined. The equation that governs the heat transfer phenomenon in the region is considered: $(-a,a)X(-b,b)\epsilon\mathbb{R}^2$where the surface $S=(-a,y)\cup(a,y)\cup(x,-b)\cup(x,b)$, which when modeled is found in [7,8] ¸ we have equation (1):

$$\frac{\partial^2 \mathrm{T}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{T}}{\partial \mathrm{y}^2}+\frac{\dot{q}}{\lambda}=0 \quad in\ -a<x<a \quad \text{and} \quad -b<y<b \quad (1)$$

Subject to the boundary conditions:

$$T = 0 \ \ in\, x = \pm a \quad and \quad y = \pm b \qquad (2)$$

the functional for equation (1) is obtained from the work of [9] as:

$$I=\int_{-b}^{b}\int_{-a}^{a}\left(\left(\frac{\partial^2 T}{\partial x^2}\right)+\left(\frac{\partial^2 T}{\partial y^2}\right)+\frac{\dot{q}}{\lambda}\right)\mathrm{T}dxdy(3)$$

Where its first variation is given by:

$$\delta I=\int_{-b}^{b}\int_{-a}^{a}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\dot{q}}{\lambda}\right)\delta\,\mathrm{T}dxdy=0 \quad (4)$$

by distributing the integration operator. Whence the functional I[T] to be minimized by equation (5). C. C. Ike, (2017):

$$I[T] = \int_{-b}^{b}\int_{-a}^{a}\left\{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 - 2\frac{\dot{q}}{\lambda}T\right\}dxdy \quad (5)$$

In order to determine the solution of Eq. (5), we select an approximate form for the dependent variable, $T(x,y)$ that satisfies the boundary conditions, by means of the Ritz, Kantorovich and Galerkin methods. We will first deal with the Ritz method. In this method we select an approximate solution $T(x,y)$which depends on n parameters, from obtain a set of functions given by the orthogonal succession $\{\phi_n(x,y)\}$convergent which is expressed in equation (6):

$$T(x,y) = \sum_{n=0}^{N} a_n\phi_n(x,y) \quad (6)$$

Where $\phi_n(x,y)$ for all values of n satisfy the boundary conditions. If $\varphi_n(x,y)$ is further considered orthogonal functions, this product of a function of x and a function on y alone is expressed in equation (7):

$$\phi_n(x,y) = X_n(x)Y_n(y) \quad (7)$$

Then $X_n(x)$ and $Y_n(y)$ will be formulated such that $X_n(x)$ satisfies only the boundary conditions in the $x - direction$and $Y_n(y)$ only the boundary conditions in the $y - direction$, so Eq. (6). is expressed by:

$$T(x,y) = \sum_{n=1}^{N} a_n X_n(x)\mathrm{Y}_n(y) \quad (8)$$

Applying the Walther Ritz method, the test function $T(x,y)$ is considered such that it satisfies the boundary conditions of the problem. For this case, one of the following series proposed by [9] can be used:

a) $T(x,y) = (a^2 - x^2)(b^2 - y^2)(a_0 + a_1x^2 + a_2y^2 + \ldots.)$
b) $T(x,y) = a_0(a^2 - x^2)(b^2 - y^2) + a_1(a^2 - x^2)^2(b^2 - y^2) + \ldots$
c) $T(x,y) = \sum_{m,n=1}^{\infty} a_{mn}\, sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)$

These must satisfy the boundary conditions of the problem, considering Eq. (a). truncated in the first term where a_0 is, for the calculation of the first Ritz approximation:

$$T(x,y) = (a^2 - x^2)(b^2 - y^2)a_0 \quad (9)$$

And substituting Equation. (9) in Equation (5)and integrating, we obtain the coefficient a_0

$$a_0 = \frac{5}{8}\frac{\frac{\dot{q}}{\lambda}}{(a^2+b^2)} \quad (10)$$

Substituting Equation (10) in Eq. (9), the solution is obtained.

$$T(x,y) = \frac{5}{8}\left(\frac{\dot{q}}{\lambda}\left(\frac{1}{(a^2+b^2)}\right)\right)(a^2 - x^2)(b^2 - y^2) \quad (11)$$

On the other hand, for the method of Leonid Vitalievich Kantorovich consists in specifying some behavior of some of the functions in some $x\ or\ y$ direction of the sets of functions that depend on these $X_n(x)$ or $\mathrm{Y}_n(y)$ used by [9] thus given the test polynomial function is given by equation (12):

$$T(x,y) = (b^2 - y^2)X(x) \quad (12)$$

Which satisfies the boundary conditions of the problem in the $y - direction$, and is considered as a first approximation. As in the previous case only now the unknown function is $Y_n(y)$. A variable exchange can also be performed by taking $X_n(x)$.

$$T(x,y) = (a^2 - x^2)Y(y) \quad (13)$$

When more precision in the calculations is desired, the degree of the polynomial proposed by [9] can be increased.

By performing the appropriate operations substituting these in Eq. we obtain:

$$I = \int_{-a}^{a} \left[\frac{16}{15} b^5 [u(x)]^2 + \frac{8}{3} b^3 u^2(x) - \frac{8}{3}\frac{\dot{q}}{\lambda} b^3 u(x)\right] \quad (14)$$

Now using the Euler-Lagrange equations:

$$-\frac{\partial F}{\partial u(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'(x)}\right) = 0 \quad (15)$$

Where F is given as:

$$F = [(b^2 - y^2)u'(x)]^2 + [-2yu(x)]^2 - 2\frac{\dot{q}}{\lambda}(b^2 - y^2)u(x) \quad (16)$$

Applying Eq. (15) and simplifying:

$$u''(x) - \frac{5}{2b^2} u(x) = -\frac{5}{4}\left(\frac{\dot{q}}{\lambda}\frac{1}{b^2}\right) (17)$$

This differential equation can be solved by two different methods, indeterminate coefficients and parameter variation.

$$u(x) = \frac{\dot{q}}{2\lambda}\left(1 - \frac{cosh\left(\sqrt{\frac{5x}{2a}}\right)}{cosh\left(\sqrt{\frac{5a}{2b}}\right)}\right) \quad (18)$$

Substituting Equation (18) in Eq. (12), we have:

$$T(x, y) = \frac{\dot{q}}{2\lambda}\left(1 - \frac{cosh\left(\sqrt{\frac{5x}{2a}}\right)}{cosh\left(\sqrt{\frac{5a}{2b}}\right)}\right)(b^2 - y^2) \quad (19)$$

Garlekin's method considers the following problem:

$$L[T(r)] = 0 \quad in \;\; R \quad (20)$$

$$B[T(r_s)] = f(r_s) ontheboundaryofS \quad (21)$$

Where L is the linear differential operator, $\left[i, e, L[T] = \nabla^2 T + \Delta T^2 + \left(\frac{\dot{q}}{2\lambda}\right)\right]$, where the function $\varphi_0(r)$satisfies the inhomogeneous part of the boundary conditions Eq. (21) and the function $\varphi_j(r)$, satisfies the homogeneous part.

$$B[\Psi_0] = f(r_s) (22)$$

$$B[\varphi_j] = 0 \;\; (23)$$

$$S_j \;\; j = 1,2, \dots., n$$

The Garlekin method for the determination of the coefficientsC_j is given by Eq. (25):

$$\int_R L[\overset{\varpi}{T}_n(r)]\phi_i(r)dv = 0 \qquad (24)$$

$$\int_R L\left[\Psi_0(r) + \sum_{j=1}^{n} C_j \phi(r)\right]\phi_i(r)dv = 0 \qquad (25)$$

Solve the problem by the method of BorísGrigórievichGaliorkin. We start from the following equation (26):

$$\int_{x=-a}^{x=b}\int_{y=-b}^{b}\left[\frac{d^2\vec{T}}{dx^2} + \frac{d^2\vec{T}}{dy^2} + \frac{\dot{q}}{2\lambda}\right]\varphi_i(x,y)\,dxdy = 0 \qquad (26)$$

Where $\vec{T}_1(x,y) = C_1\varphi_1(x,y)$ and $\varphi_1(x,y) = (a^2 - x^2)(b^2 - y^2)$replaced in eq. (26). Equation (27) is obtained:

$$T(x,y) = \frac{5}{8}\left(\frac{\frac{\dot{q}}{2\lambda}}{a^2+b^2}\right)(a^2 - x^2)(b^2 - y^2) \qquad (27)$$

The exact solution using Fourier series is given by [1,10,11]:

$$T(x,y) = \frac{\dot{q}}{\lambda}\left[\frac{a^2-x^2}{2}\right] - 2a^2\sum_{n=0}^{\infty}\frac{(-1)}{B_n^3}\frac{cosh\left(B_n\frac{y}{b}\right)*cosh\left(B_n\frac{b}{a}\right)}{cosh\left(B_n\frac{b}{a}\right)} \qquad (28)$$

Where:

$$B_n = \frac{(2n+1)\pi}{2}$$

In order to compare the results obtained by numerical solution of these methods presents Table no 2.

Taking the following assumptions for our case.

$$x = 0, y = 0, a\; x = 1\;\&\; y = 1\;, q` = 1, \lambda = 1$$

Table (2) shows the values obtained by the methods, as well as the percentage error of each method.

Table no 2: Errors made when using the methods.

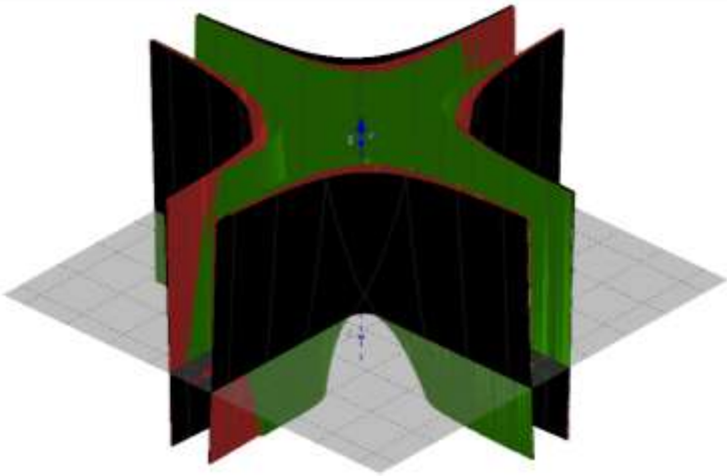
Method	True value	Calculated value	Percentage error
Ritz	***0.293***	0.3125	6.655290102389078498
Kantorovich	***0.293***	0.3029854	3.4124709897610921
Galerkin	***0.293***	0.3125	6.655290102389078498

III. Results and Discussion

In the presented methods the one of Walther Ritz is a little laborious in mathematics so it is also necessary to be careful in the integration, in order not to have losses in the solution, the one of BorísGrigórievichGaliorkinwhen one finishes to integrate one drinks to use the equations of Euler-Lagrange to solve the ordinary differential equation that these can be coupled. But as the expert knows how it behaves in one direction, he proposes a test function and finds the other by this.

The Graph no 1 shows the behavior of the solutions in an interval of$[-7,7]$ X $[-7,7]$ X $[-7,7]$.

Graph no 1: Shows the temperature distribution of the Ritz, Kantorovich, Galerkin methods and the errors in the calculations in the solution of the application.



IV. Conclusion

The Kantorvich method is a simple method, but the mathematical procedure is complicated, compared to the other two, so it can be concluded that the other two can be used with more terms to improve the approximation, because its development is simple to obtain a better approximation.

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