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A Review paper on Quantum harmonic oscillator and its applications in finance

Atman Bhatt[1], Ravi Gor[2]
[1]*Research scholar, Department of Applied Mathematical Science, Actuarial Science and Analytics, Gujarat University*
[2]*Department of Applied Mathematical Science, Actuarial Science and Analytics, Gujarat University*

Abstract: *The quantum harmonic oscillator is a quantum mechanical counterpart to the classical harmonic oscillator. It's also one of the few quantum-mechanical systems for which a precise, analytical solution exists. It is one of the most significant model systems in quantum mechanics because a given smooth potential can generally be represented as a harmonic potential at a vicinity of a stable equilibrium point. The tiny vibrations in a diatomic molecule are classically viewed using a quantum harmonic oscillator model, but the representation is universal in the sense that it may be extended to a range of other circumstances in physics and beyond. The main purpose of this paper is to review applications of quantum harmonic oscillator model in financial mathematics.*

Key word: *Quantum Harmonic Oscillator, Diatomic molecule, Harmonic potential, Smooth Potential.*

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I. Introduction

Finance theory is based on instrument pricing such as stock and option pricing. Finance community are facing many issues which have unknown analytical solution. As a result, the use of computer simulations to solve these challenges has exploded. Many computational finance problems are highly computationally complicated and take a long time to solve on classical computers. Option pricing becomes much more complicated as a result of the need to react to quickly changing markets.

To take advantage of inaccurately priced stock options, for example, the computation must be completed before the next shift in the stock market, which is almost constantly changing. As a result, the financial community is constantly looking for solutions to the performance challenges that develop as a result of price options. This has led to financial study using alternate computing techniques. Quantum computing is one of these options. Computing, like physics models, has progressed from classical to quantum. Quantum computers have been demonstrated to outperform classical computers in the simulation of quantum mechanics as well as various other algorithms such as Shor's factorization method and Grover's quantum search algorithm, making them a promising topic for research in computational finance. The concept of bringing quantum mechanics to finance is not new: some well-known financial problems can be directly described in quantum mechanics. The Black-Scholes-Merton formula, for example, can be transferred to the Schrodinger equation, simulating the arbitrage linkages that led to its formulation. Even the entire financial market can be modelled as a quantum process, in which important financial quantities like the covariance matrix emerge spontaneously. Many financial problems can be described as optimisation problems. These are problems that are difficult for classical computers to solve, but which can be solved naturally employing quantum optimization techniques.

This topic has risen in popularity in recent years, partly in response to the commercial availability of quantum computers. Nowadays many physical, mechanics and computer entities are being quantized and used for the betterments and research purposes. One important phenomenon of classical physics say, harmonic oscillator is a system that fluctuates away from equilibrium (like a pendulum) but has a restoring force that brings it back to it. The quantum analogue, a quantum harmonic oscillator, is a system that is displaced from equilibrium and has a restoring force, but it differs from the classical system in several ways, including the quantization of its energy levels (discrete). Here we want to focus on the quantum-mechanical analogue of the classical harmonic oscillator named quantum harmonic oscillator which is one of the most important model systems in quantum mechanics. It is one of the few quantum-mechanical phenomena for which an exact analytical solution is known.

Quantum Harmonic Oscillator:

In nature, a quantum harmonic oscillator is a micro-physical system with a mathematical structure derived with the help of Bohr correspondence principle from the classical harmonic oscillator. There are numerous micro-physical systems like this. The diatomic molecule provides an example, as long as the energy is not too high and there is no rotation. It is frequently much easier to evaluate a system like a diatomic molecule in terms of how it moves as a whole—such as vibrate or rotate—rather than breaking it down into its numerous constituent particles. Collective motion refers to the overall motion of the system. The transition from classical to quantum entities is accomplished by substituting classical quantities such as position x and momentum p with the corresponding quantum mechanical operators Q and P, as per the quantization principles. The Hamiltonian H, a quantum energy operator, is then derived from the classical energy formula

$$E^{classical} = \frac{p^2}{2\mu} + \frac{1}{2}kx^2$$

by making the replacements $E^{classical} \to H, x \to Q, p \to P$. As a result, the Hamiltonian is thought to represent the energy operator for the quantum harmonic oscillator and to express k in the terms of ω. Because $\omega = \sqrt{\frac{k}{\mu}}$ (System follows oscillations with an angular frequency). After making the replacements, we will get quantum energy formula:

$$H = \frac{P^2}{2\mu} + \frac{\mu}{2}\omega^2 Q^2$$

The operators P and Q are postulated to satisfy the Heisenberg relations $[Q, P] = QP - PQ = i\hbar\mathbb{1}$ and $[P, P] = [Q, Q] = 0$ where $\mathbb{1}$ is the unit operator and $\hbar = \frac{h}{2\pi}$, where h is the Planck's constant. The constants μ and k are mass of the particle and kinetic energy constant. In the classical case the energy $E^{classical}$, the momentum p, and the position x are real numbers whereas in the quantum-mechanical case the quantities are represented by the self-adjoint operators H, P, and Q, respectively. The constants μ and ω are characteristic of the particular physical system.

The Hamiltonian based on quantum harmonic oscillator, with $\frac{k}{\mu} = constant$, provides a satisfactory description in a limited energy range. Since vibrating diatomic molecules are the only harmonic oscillators described by the Hamiltonian here, only a finite number of the energy eigenstates $|E_n\rangle$ are physically important. The quantum harmonic oscillator has a lowest energy value $E_0 = \frac{\hbar\omega}{2}$, known as the zero-point energy. This feature, particular to quantum mechanics, is to be contrasted with the classical convention that the minimum energy of the oscillator is zero.

Physical quantum harmonic oscillators occur in nature in many different forms, in particular as diatomic molecules such as O_2, HCl, and CO. Each of these oscillators has different values of the system constants k and μ, resulting in a different value of angular frequency ω. But otherwise, the Hamiltonian of each of the vibrating diatomic molecules has the same form written above, leading to equal spacing between energy levels. The vibrating diatomic molecule is one such physical system that can be approximately described as a quantum harmonic oscillator. From energy loss experiments, for diatomic molecules the energy levels are found to be equally spaced within a certain range of data. Thus, within this range, diatomic molecules can be described as quantum harmonic oscillators.

A quantum harmonic oscillator model is generally used to represent the tiny vibrations in a diatomic molecule, but the description is universal in the sense that it can be expanded to a range of other cases in physics and even beyond, as we mentioned before. A new study shows that the restoring force in a vibrating quantum harmonic oscillator seems to give a decent approximation of the market force that returns a volatile stock to equilibrium.

Diagram:

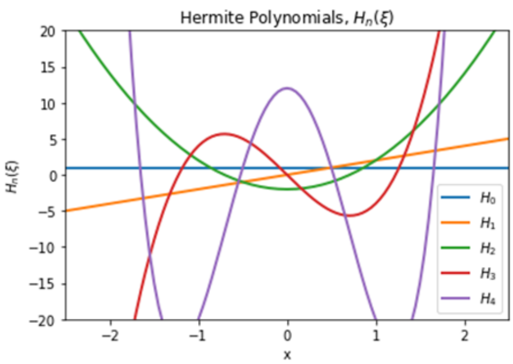

Figure 1

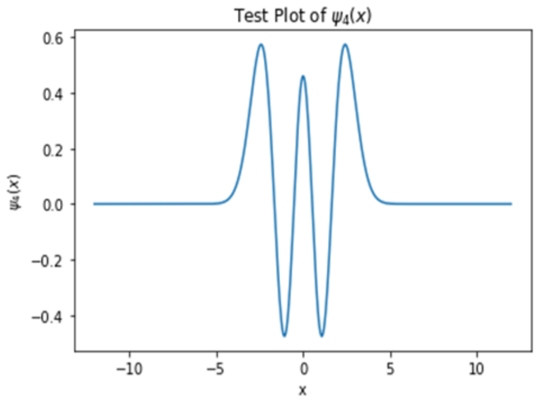

Figure 2

Figure -1 showcase the Eigen-function based on the time independent, quantum harmonic oscillator where the normalized stationary states for the oscillator are given in the terms of Hermite polynomials,

Figure -2 elaborates the Eigen-states and the probability density distributions of the quantum harmonic oscillators for the Eigen-states n=4. Figure-3 gives the exact difference about classical and quantum harmonic oscillator where the classical harmonic oscillator is denoted by the orange colour and its quantum counterpart is denoted by the blue colour.

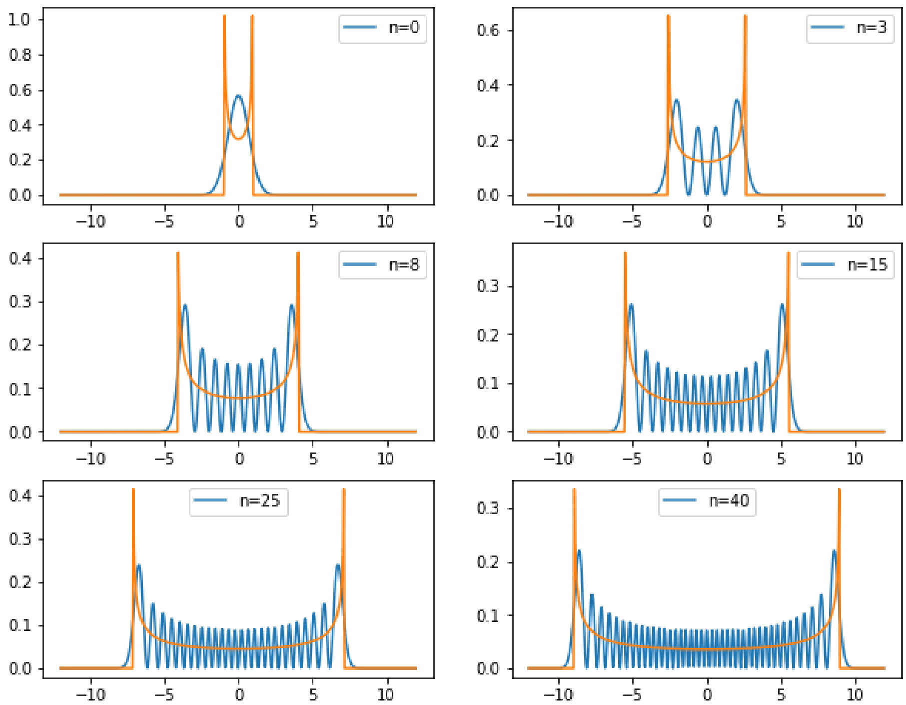

Figure 3

From the diagram one can easily see that, at the centre of the well, the probability density distributions for a quantum oscillator in the ground low-energy state are largest. It is expected that the particle spent its most of the time to oscillate at the centre of the well in order to be detected there with the greatest likelihood. This is in contrast to the behaviour of a classical oscillator, in which the particle spends the majority of its time at the turning points travelling at relatively slow speeds.

Some important axioms regarding quantum harmonic oscillator:

- Allowed energies are discrete and uniformly spaced for quantum harmonic oscillator and the spacing between the energies are as same as the Planck's energy quantum.

- The ground state energy is always greater than zero which means a quantum oscillator is never at rest like its classical counterpart, even at the bottom of a potential well.
- For high quantum numbers, the motion of a quantum oscillator becomes more similar to the motion of a classical oscillator as per the Bohr's correspondence principle.
- The stationary states (states of definite energy) have nonzero values also in regions beyond classical turning points. When in the ground state, a quantum oscillator is most likely to be found around the position of the minimum of the potential well, which is the least-likely position for a classical oscillator.
- The probability of finding a ground-state quantum particle in the classically forbidden region is about 16%.
- Eigen-functions of quantum harmonic oscillator are nothing but the Hermite polynomials with degree depends on the oscillations.

II. Literature Review

Ye and Huang (2008) proposed a model to explain persistent economic fluctuations was based on quantum mechanics, while Frisch's classical damping oscillator model has failed to do so successfully. Since the same stock has a price range rather than a set price at different times, this is predicated on the concept that the value might be a wave packet that determines the probability of each price. In this situation, the market is viewed as a device capable of determining value and determining a price. The numerical simulation findings are then used to qualitatively explain long-term stock market changes.

Ataullah (2009) shows how quantum mechanics' complementarity principle relates to stock market prices, and how the wave function it generates leads to a probability density that is very compatible with the FTSE All Share Index's returns. Our research reveals that the probability density of stock market returns is strongly leptokurtic, with a minor negative skewness.

Cotfasa and Cotfas (2013) look into an extension that is actually associated to the quantum harmonic oscillator. Using a finite oscillator and the Harper functions, a finite-dimensional version is created. This reduced model retains the key qualities of the continuous one, but instead of series and integrals, it uses finite sums.

Meng et al. (2016) construct an econophysical structure for the stock market based on quantum mechanics' physical concepts and mathematical structures, in which we analogously map massive numbers of single stocks into a reservoir consisting of many quantum harmonic oscillators, and their stock index into a typical quantum open system—a quantum Brownian particle. In Heisenberg's uncertainty principle of quantum mechanics, the irrationality of stock transactions is quantitatively treated as the Planck constant in a similar manner. Furthermore, he uses the quantum Brownian motion model to analyse real stock data from the Shanghai Stock Exchange of China and investigate fat-tail phenomena and non-Markovian behaviours of the stock index, interpreting and studying the limitations of the classical Brownian motion model for the efficient market hypothesis from a new perspective of quantum open system dynamics.

Gao and Chen (2017) offer a financially interpretable quantum model to examine the probability distributions of stock price returns. The dynamics of a quantum particle are likened to the movement of a stock market. Then, using the wave functions that evolve according to the Schrodinger equation, the probability distributions of price return can be determined. In this study, instead of using a harmonic oscillator, a quantum anharmonic oscillator is used on the stock in the liquid market. With the addition of mixed-state and multi-potential, proposed quantum model effectively recreates leptokurtic price return distributions. As a special case of the illiquid market, the trend following dominant market is examined, in which the price return follows a bimodal distribution.

Ahn et. al. (2018) demonstrate that the quantum harmonic oscillator model outperforms traditional stochastic process models, e.g., geometric Brownian motion and the Heston model, with smaller fitting errors and better goodness of fit statistics. The solution of the Schrodinger equation for the quantum harmonic oscillator shows that stock returns follow a mixed distribution, which describes Gaussian and non-Gaussian features of the stock return distribution. In addition, they provide an economic rationale of the physics concepts such as the eigenstate, eigenenergy, and angular frequency, which sheds light on the relationship between finance and Physics literature.

Oduro et. al. (2021) designed study to obtain the energy eigenvalues and the corresponding Eigenfunctions of the Quantum Harmonic oscillator through an alternative approach. They recover the Schrodinger Equation together with its eigenvalues and eigenfunctions of the quantum harmonic oscillator via the use of Gram Schmidt orthogonalization process in the usual Hilbert space. Significantly, it was found that there exist two separate sequences arising from the Gram Schmidt Orthogonalization process; one in respect of the even eigenfunctions and the other in respect of the odd eigenfunctions. In the case of a strike price that is not too far from the current price of the underlying asset, the Black-Scholes model predicts the observed pricing for options quite well. A proper modification of the coefficients in the Black-Scholes equation can yield some helpful extensions.

Xiangyi et al. (2022) explores the behaviour of stocks in daily price-limited stock markets. In a quantum spatial-periodic harmonic oscillator potential well, the stock price is considered to be oscillating and damping. With the energy band structure of the model in question, a sophisticated non-linear relationship between the volatility and trading volume of a stock is statistically calculated, including inter-band positive correlation and intra-band negative correlation. The efficiency of price limits is re-examined, with our quantum model being used to investigate some observable aspects of price-limited stock markets in China. The Financial Times-Stock Exchange (FTSE) All Share Index's instantaneous return is modelled as a frictionless particle travelling in a one-dimensional square well with a non-trivial probability of tunnelling through the well's retaining walls.

III. Conclusion and Future scope

Because the harmonic oscillator is just a lowest order estimation of an arbitrary binding potential, the quantum harmonic oscillator is essential to every problem regarding physics and finance, involving quantum degrees of freedom in a potential well. Quantum harmonic oscillator (QHO) involves square law potential (x^2) in the Schrodinger equation and is a fundamental problem in quantum mechanics. The quantum harmonic oscillator model can be solved using a variety of traditional methods, including:

(i) analytical methods involving Hermite polynomials,
(ii) algebraic methods involving ladder operators, and
(iii) approximation methods involving perturbation, variation method, semiclassical, and other techniques.

Beyond the small diatomic molecule, the quantum harmonic oscillator has far-reaching implications. It is the basis for comprehending complicated modes of vibration in larger molecules, the mobility of atoms in a solid lattice, and many more applications in heat capacity theory, stock distribution in financial mathematics, supply and demand, and so on. Only at the lowest levels, where the potential is a reasonable approximation of the "mass on a spring" type harmonic potential, are energy spacings identical in real systems. The anharmonic terms that exist in a diatomic molecule's potential can be used to map the detailed potential of such systems. Finally, we can say that the quantum harmonic oscillator model can be used in various issues regarding financial mathematics because of its ample applications and useful axioms. Even many modifications can be done in quantum harmonic oscillator as per the demands of the respective problems. Formulations can also be made for the stock distribution with the help of quantum harmonic oscillator model which can give better results for goodness of fit tests.

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