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Derivation Of Darcy's Law Using Homogenization Method

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Abstract
Derivation of Darcy's law using Homogenization Method
The aim of this work is to show homogenization techniques for solving complex phenomena in heterogeneous media. First, we begin with a brief description of a porous medium and some flow equations in a porous medium. Then, we will study the homogenization theory whose main objective is to use this homogenization to find Darcy's law. As a reminder, Darcy's law is a law of flow of water in a porous medium obtained by Henry Darcy during his experiments which enabled him to find the connection between the flow of water through the sand and the pressure drop of water. We are then interested in Stokes equations considered as equations in the microscopic state and using method indicated above, we will solve this equation in order to find Darcy's monophasic and diphasic law.
Keywords*: Derivation, Darcy's law, Homogenization.*

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I. Introduction

To be able to study the physical phenomena intervening in nature, one firstly needs a model allowing to determine equa- tions governing these phenomena, and then one needs to be able to compute the solutions of these equations. However, sometimes the resolution of these problems is very difficult because of the complexity of problem, for example when the environment in which the physical phenomenon is studied, presents itself a very great complexity. We focus on homog- enization method to solve these problems. The aim of the homogenization theory is to obtain a homogeneous (simple) approximation of a medium described by microscopic properties that are supposed to be very heterogeneous. In this work we are interested in periodic homogenization method. We will use latter to determine monophasic Darcy's law. This well-known Darcy?s law is one of the most important equations used to model the fluid flow through a porous medium. Equation shows a relationship between the superficial velocity or Darcy velocity and the pressure gradient that was first experimentally observed by Henry Darcy in 1855-1856. We will try to use this same method to determine the Darcy- Muskat law, which is a two-phase law applied to the flow of two fluids through a heterogeneous porous medium. This interest is due to the wide range of applications of such a phenomenon for example: study of subsoils (diffusion of oil in porous media), properties of composite materials, etc. This paper is organized as follows : First we focus on generalities by describing porous media, their characteristics and properties. We also introduce the theory of homogenization. The other part will be devoted to the determination of the Darcy equation governs the fluid flow in porous media using periodic homogenization. The following part will be the subject of presentation of equations that govern two-phase flows in porous media. We will end with conclusion.

Generality

A porous medium is a medium consisting of a solid matrix and its geometric complement, porous space, the latter may be occupied by one or more fluids. Porous media are ubiquitous in our natural environment as well as in technological and industrial systems. The porous space is called monophasic when it is occupied by a single fluid see figure 2.1 (red is fluid occupying pores and white part is solid). It is diphasic when it is occupied by two fluids see figure 2.2 (This figure contains the solid part with white color and two fluids occupying the pores, one with red color and the other with blue color).

In general, a volume of porous material can be naturally occupied by a fluid when it has pores. Pores are spaces that are not occupied by solid constituents. In this work we are interested in the fluid flow in a porous medium consisting of a single solid component characterized by porosity, saturations and permeabilities.

Homogeneous medium.

A homogeneous medium is a medium where in every point of it, the characteristics are identical. But a heterogeneous medium is a medium composed of several identifiable constituents, it is a chemical or physical system formed of several phases. We can see in the figure 2.3 starting from the heterogeneous medium to a homogeneous medium.

Porosity.

The porosity φ of a material represents the pore density that can be occupied by a liquid or gaseous fluid. It is expressed by the ratio of void volume to the total volume occupied by material:

Where v_p and v_p are respectively the void volume and the total volume occupied by the material. The porosity is between

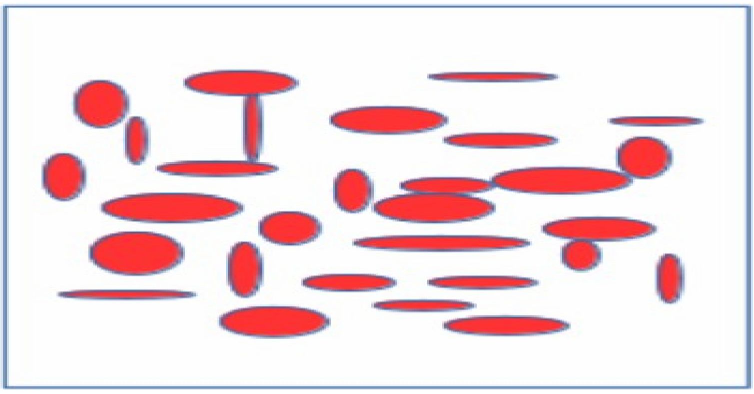

Figure 1: Porous medium occupied by a single fluid

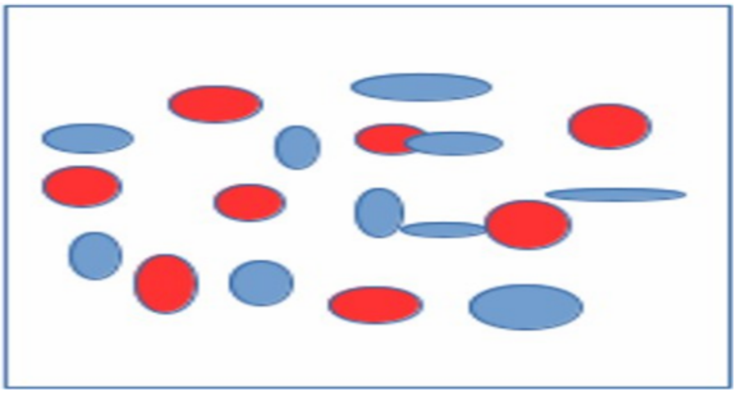

Figure 2: Two-phase porous medium

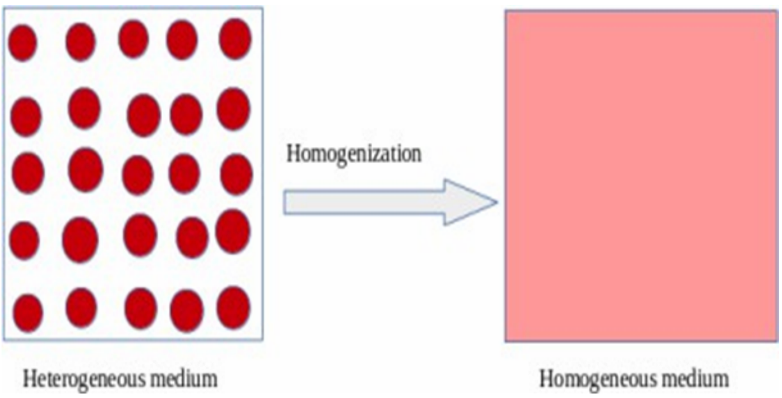

Figure 3: Diagram showing a homogenized heterogeneous medium.

0 and 1

$0 \leq \varphi \leq 1$

For example in saturated medium with water, the gaseous phase is completely absent. The maximum proportion of water in the material is therefore equal to the porosity of this material. The voids in the material can be connected to each other, this is called open porosity. If the pores are without communication, we talk of closed porosity.

Intrinsic permeability κ

Permeability is a physical characteristic that represents the ease with which a material can transfer a fluid through a connected network. This characteristic of porous medium is related to the shape of the grains and to the communication between the pores. It is independent of the fluid characteristics and depends only on the structure and the connectivity of the pores.

Isotropic medium

An isotropic medium is a medium that has the same physical characteristics in all directions; the macroscopic properties at one point do not depend on direction.

The permeability coefficient K, or the intrinsic permeability κ, are scalar coefficients if the porous medium is isotropic.

Anisotropic medium

An anisotropic medium is a body whose properties vary according to the direction.

When the medium in three-dimensional space is anisotropic, the coefficient of hydraulic conductivity is defined by a symmetric tensor of the form :

By locating in the coordinate system whose axes are the directions for which the flow is parallel to the load gradient, the conductivity tensor is reduced to its diagonal components :

Two-phase description

Consider a flow of two immiscible fluids through a porous medium such as flow of water and oil. u_α and P_α are re- spectively the speed and pressure of phase α. $\alpha \in \{w, O\}$; og where w represents water phase and O represents oil phase.

saturation

In a two-phase porous medium, the saturation S_α of phase α is defined by the ratio of volume of the phase α to total volume.

Where V_α is the volume of phase α and V_t is the total volume. The sum of saturations is equal to 1.

$$S_w + S_o = 1 \quad (4)$$

density

Density is a physical quantity that characterizes the mass of a material per volume unit. The density ρ_α of the phase α is defined by the ratio of the mass by the volume of this phase. with m_α and V_α are respectively mass and volume of phase α.

Darcy law

In the framework of his experiments (see figure 4) to improve the quality of filters used for purification of water supply of Dijon in France, Henry Darcy was the first to observe in 1856 the relationship between the flow rate of steady-state water flow through a porous medium of section S and length L under the effect of a load difference a*h*. Furthermore, the coefficient was dependent on the type of sand used. From these observations, he got the following law :

with S the section area of the sand layer, K the permeability coefficient depending on the nature of sand used and Q is the flow of filtered water. a*h* is the pressure drop of the water between top and bottom of sandy mass, L the thickness of the sand mass and $\frac{ah}{L}$ is the hydraulic load gradient . The hydraulic load in one point expressing in a general way as follows :

with P pressure, ρ density, *g* is gravity acceleration and *z* depth. If one divides the flow Q by section S, one obtains a speed, called Darcy velocity u corresponding to a fictitious velocity since it supposes that the whole surface, including the matrix, participates in the flow.

This law has been established for a monophasic flow in a saturated porous medium. In modern notation, this is expressed, in a local form, by the differential relation

Here $\frac{\partial P}{\partial x}$ is pressure gradient in the direction of flow, K is the permeability and μ is dynamic viscosity of fluid. Darcy's law is now generalized to incompressible fluids by expressing it according to the intrinsic properties of porous medium and the fluid. This law can be generalized to unsaturated media and in 3 dimensions:

where *u* is Darcy velocity vector and K represents permeability and depends on the characteristics of the medium and characteristics of fluid. Absolute intrinsic permeability K can be defined by relation :

Darcy's law is now generalized to compressible fluids by expressing it according to the intrinsic properties of the porous medium and fluid.

When a porous medium contains two immiscible fluids, the flow of one of these fluids is slowed by the presence of other. Darcy's law called Darcy-Muskat written for each phase α :

with $\alpha \in \{o, w\}$ where index *w* represents water phase and *o* represents oil phase. Each phase has its own saturation s_α as well as its pressure p_α, K is a tensor of absolute permeability, λ_α is the relative permeability of phase α and satisfies $\lambda_\alpha(0)$
= 0, ρα is the density of phase α and *g* is gravity. Absolute permeability tensor K is expressed as :
Where $k_{r\alpha}$ is relative permeability of phase α.

This Darcy-Muskat law makes it possible to generalize Darcy's law to multiphasic cases. It expresses the mass fluxes of different phases by introducing the notion of relative permeability.

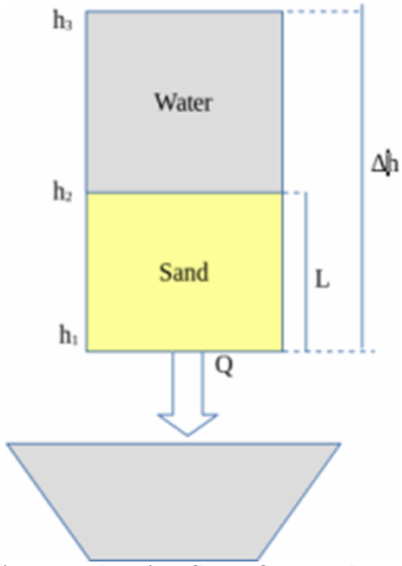

Figure 4: Diagram showing flow of water through the sand

We started from mass conservation equations and momentum equation to deduce Navier-Stokes equations. Consequently, for the case of a viscous homogeneous incompressible fluid, it has been considered to deduce Stokes equations. We will see in the next chapter, the general description of homogenization theory that will be applied later to derive Darcy laws.

II. Homogenization Of Stokes Equations

In this section we will present Darcy's derivation from the Stokes problem. We will use homogenization method.

Stokes problem

Consider a moving fluid in a periodic, open domain, connected $\Omega^s \subset R^3$. Let us denote by Ω_F the fluid phase, Ω_S the solid phase and a continuous liquid-solid interface in pieces $\Gamma = \Gamma_F \cup \Gamma_S$ where Γ_S and Γ_F are the boundaries of solid and liquid phases respectively see figure 5. Suppose the fluid is viscous and incompressible homogeneous, we model the flow with Stokes equations :

$$\begin{cases} \nabla P^\epsilon - \epsilon^2 \mu \Delta u^\epsilon = f, & in \quad \Omega^\epsilon \\ \nabla . u^\epsilon = 0, & in \quad \Omega^\epsilon \\ u^\epsilon = 0, & in \quad \Gamma_S \end{cases}$$

where u^s (respectively P^s) is velocity (respectively pressure) of fluid. f = (f_1, f_2, f_3) denotes force density and $\mu > 0$ the dynamic viscosity of fluid supposed constant and independent of s. The viscosity μ has been multiplied by s^2 because our system is linear and we can always replace u^s with s^2u^s.

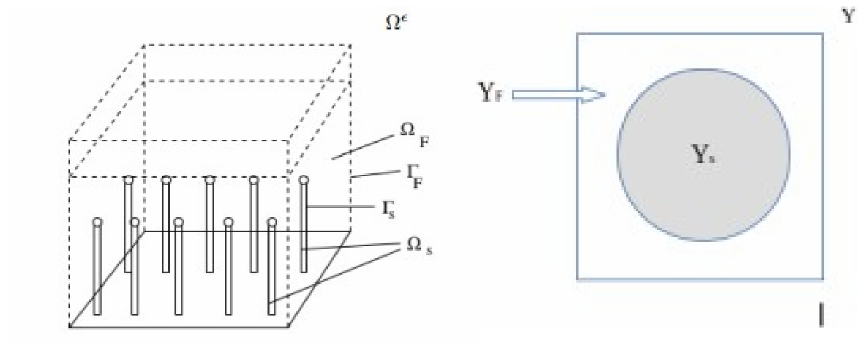

Figure 5: Porous medium containing single fluid

We note s the ratio of the pore diameter to the overall size of the porous medium: this is the small parameter of our asymptotic analysis since the pore size is generally smaller than the characteristic length of the reservoir. So let's take the pore diameter d = 2r where r is the radius and choose L the size of the porous media, then $s = \frac{d}{L} \ll 1$.

Since the domain Ω^s is assumed to be periodic with period s, consider Y = (0, 1)^3 the unit cell of Ω^s . Let x be a macroscopic variable and define a microscopic variable ($x \in \Omega^s$ then $y = \frac{x}{s} \in Y$)

Asymptotic development

We will perform an analysis using asymptotic expansion u^s such that $\lim u^s = u$ with u solution of a homogeneous

The results obtained in this chapter have highlighted the interest of the periodic homogenization technique to solve prob- lems by from the microscopic state to the macroscopic state Cf. [Cha13]. We apply this homogenization method to the Stokes problem for a monphasic flow that allowed us to determine the Darcy's law. We will see in the next chapter, the application of this technique for two-phase case.

III. Periodic Homogenization Method Applied To The Darcy Diphasic Law

We propose in this chapter to present implementation of periodic homogenization method for the flow of two fluids in a porous medium. We will show, using periodic homogenization method, that the viscous flow of two immiscible fluids is governed by macroscopically coupled Darcy laws.

Choice of scales

We assume that Ω^{*s} be a periodic domain occupied by two fluids water and gas. Ω^*_w part containing water(w) and Ω^*_g the one occupied by gas(g). Interface water-gas noted Γ^* moves slowly. As we did in the previous chapter, we recall the scale separation, we choose as macroscopic space variables x^* and microscopic y^* are completely independent, which amounts to considering the characteristic length of the elementary cell l much smaller than the length characteristic of the global structure L :

$$l \quad L$$

We also introduce the perturbation parameter s which represents the ratio between the characteristic size at microscopic scale and the characteristic size at macroscopic scale :

So, scale separation becomes : l

$$l$$

We use an approach based on an adimensionalization of equilibrium equations using reference variables. Then, the dimensionless numbers introduced must be connected to s, chosen as the disturbance parameter of the problem, in order to perform the asymptotic development of equations.

Microscopic scale description

We assume that the porous medium is rigid, and that the two fluids (water and gas) are viscous, Newtonian and incom- pressible. On the other hand, we will also assume the inertial effects can be neglected and the two fluids (liquid and gas) are laminar movement, permanent and isothermal. At the pore scale, Reynolds numbers associated with the two fluids are supposed to be small, that one writes :

$R_{ew} \leq O()$ *and* $R_{ew} \leq O()$

At the level of elementary cell, the liquid is governed by Stokes equation (incompressible laminar flow) :

$$\begin{cases} \mu_w^* \Delta u_w^* &= \nabla P_w^* \;\; in \;\; \Omega_w^* \\ \nabla . u_w^* &= 0 \;\; in \;\; \Omega_w^* \\ u_w^* &= 0 \;\; in \;\; \Gamma_{ws}^* \end{cases}$$

where denotes the liquid-solid interface in the elementary cell of Ω_*.
In the same way, Stokes equation for gas is written :

$$\begin{cases} \mu_g^* \Delta u_g^* &= \nabla P_g^* \;\; in \;\; \Omega_g^* \\ \nabla . u_g^* &= 0 \;\; in \;\; \Omega_g^* \\ u_g^* &= 0 \;\; in \;\; \Gamma_{gs}^* \end{cases}$$

where denotes the gas-solid interface.
The boundary conditions at the liquid-gas interface are given by the continuity of pressure on :

$$(\gamma_g^* - \gamma_w^*).n^* = P_c^*.n^* \;\; in \;\; \Gamma_{wg}^*$$

where , , and respectively denote constraints of liquid and gas, , is capillary pressure , and is unitary normal at the interface
As moves slowly, we note :

.n0() in

.n0() in

With

.n .n in

We introduce the dimensionless magnitudes of the problem: where the variables indexed by r are the reference variables, and the new variables that appear (without star) are dimensionless.
By introducing these dimensionless variables 67 in equations 61 - 66, we obtain the dimensionless problem for the liquid

We introduce the dimensionless magnitudes of the problem

by introducing variables indexed by r which will be the reference variables and the new variables which appear (without a star) are dimensionless. By introducing them into the equations above, we will obtain the dimensionless problem for the liquid

$$\begin{cases} \mu_w \Delta u_w &= \mathcal{J}_w \nabla P_w \ \ in \ \ \Omega_w \\ \nabla . u_w &= 0 \ \ in \ \ \Omega_w \\ u_w &= 0 \ \ in \ \ \Gamma_{ws} \end{cases}$$

and similarly we get for the gas :

$$\begin{cases} \mu_g \Delta u_g &= \mathcal{J}_g \nabla P_g \ \ in \ \ \Omega_g \\ \nabla . u_g &= 0 \ \ in \ \ \Omega_g \\ u_g &= 0 \ \ in \ \ \Gamma_{gs} \end{cases}$$

The chosen approach is based on an adimensionalization of equilibrium equations written in local form, so as to reveal dimensionless numbers governing the problem. Then, the asymptotic development of equations leads to the desired homogenized macroscopic model. The generalized Darcy equation was found by periodic homogenization, connecting the fluid velocity to the pressure gradient.

IV. Conclusion

During this Master thesis, we have been interested in describing equations describing the flow of an incompressible vis- cous fluid. We started from the microscopic equations of Stokes. The objective of this work was to obtain the macroscopic equations of fluid flows in a porous medium. These macroscopic equations, that describing the monophasic flow is Darcy's law and the other one describing the two-phase flow is Darcy-Muskat law. To achieve this goal, we started from homog- enization theory of periodically heterogeneous medium. This part is an important tool for solving microscopic equations describing natural phenomena. We defined two variables, one is microscopic variable and the other one is macroscopic. These last were related to the s perturbation parameter, the ratio between the microscopic and macroscopic scales. From a thermal equation, the implementation of asymptotic expansion allowed us to find homogenized problem as well as the homogenized coefficients. Some properties for two-scale convergence have been reported closing this part. We used an asymptotic development approach to determine Darcy's law from Stokes equations. Finally, the dimensional analysis of flow equation of two fluids governed by Stokes equations, allowed us to naturally highlight the relevant dimensionless numbers characterizing the problem. The law of Darcy-Muskat was found by the same method previously.

Since the description of our work is focused only on the periodic heterogeneous domains and limited to two scales, among the perspectives that we envisage later in order to enrich this work, it would be interesting to expand for the heterogeneous materials, whose behaviors local mechanics may be nonlinear or time dependent. In addition, it would be interesting to use numerical methods to simulate.

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Notations

κ :intrinsic permeability
ρ : density (kg/m^3)
μ : dynamic viscosity (kg/m.s) g : gravity (m/s^2)
Q : outflow (m^3/s)
S : Flow section (m^2) h : height (m)
L : Porous medium length (m) K : Hydraulic conductivity(m/s)
Δh : Difference of heights (m)
u : Darcy velocity (m/s)
ρ_α: density of phase α (kg/m^3)
S_α: saturation of α
μ_α: viscosity of phase α ($kg/m.s$)
φ: Porosity
λ_α: mobility of phase α λ: total mobility ,
$K_{r\alpha}$: relative permeability of phase α , P_α: pressure of phase α ($kg/m.s^2$) , P_c: capillary pressure ($kg/m.s^2$),
u_α: Darcy velocity of phase α (m/s),
q_α: source term α,
f_α: flow fraction of the phase α.

∇ : gradient
$\nabla.$: divergence
a : Laplacian
D : particular derivative
Ω : Domain of porous medium
Γ : boundary of domain
v : volume
v_t : total volume
τ : Shear constraint t : time
I_3 : identity matrix
σ : constraint tensor
A^s : differential operator $ç_w$: constraint of water $ç_w$: constraint of gas
t a : transposed of Laplacian operator
Tr : trace