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Developed Operations On *Β*-Open Subsets

Dr. Mansour Samir Elsayed Elseidy
College of Health Sciences in Al-Leith, Department of Public Health
University of Umm Al-Qura, Makkah almokaramh, Kingdom of Saudi Arabia

Abstract
Considering the notion of β-open set, we insert and study topological characteristics of β-closure set, β-interior set, β-interior point, β-border, β-frontier and β-exterior, β-accumulation points, β-derived set,. The relationships between β-closure set (β-accumulation point, β-interior point, β-derived set, β-border, β-frontier and β-exterior) and pre-closure sets (pre-accumulation point, pre-derived set, pre-border, preinterior point, pre-frontier and pre-exterior) are obtained.

Keywords: *β-closure set, β-interior set, β-accumulation points, β-border set, β-derived set, β-interior points, β-frontier set and β-exterior set*

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I. Introduction

The concept of pre-open set was obtained by Mashhour & et al. [1]. In [2], Young Bae Jun et al. Studied topological properties of pre-accumulation points, pre-derived sets, pre-interior and pre-closure of a set, pre-interior points, pre-border, pre-frontier and pre-exterior by considering the notion of pre-open sets.

II. Basic ideas

The concept of pre-open set was obtained by Mashhour & et al. [1]. In [2], Young Bae Jun et al. Studied topological properties of pre-accumulation points, pre-derived sets, pre-interior and pre-closure of a set, pre-interior points, pre-border, pre-frontier and pre-exterior by considering the notion of pre-open sets. The connotation of β-open set was inserted by Abd El-Monsef & et al. [3]. In this work, we interest new characteristics of β-closure of a subset. Furthermore, we interest the connotation of β-accumulation and β-interior points of a subset. Similarly, the notion of β-derived, β-interior, β-border, β-frontier and β-exterior of a subset are introduced by taking the concept of β-open set. We provide relationships between β-closure set (resp. β-derived set, β-accumulation point, β-interior point, β-border, β-frontier, and β-exterior) and pre-closure set (resp. pre-derived set, pre-accumulation point, pre-interior point, pre-border, pre-frontier and pre-exterior).

III. Preliminaries

In our manuscript, (M, σ) and (N,ψ) mean topological spaces. $K \subseteq M$ is called β-open ([3], [4]) (resp. pre-open [5] and semi-open [6]) if $K \subset K^{-\circ-}$ (resp. $K \subset K$ and $K \subset K$). The complementary of β-open set [3] (resp.an pre-open set [5] and a semi-open set [6]) is said to be β-closed set (resp. a pre-closed set and a semi-closed set). We symbolize the class of β-open sets (resp. pre-open sets and semi-open sets) of (M, σ) by σ^{β} (resp. σ^{p} and $\sigma^{s)}$. Clearly, we obtain the following diagram.

Example 1 ***:***
Consider the topology σ = {ϕ,M,{r},{t,z},{r,t,z}} on the set M = {k,r,s,t,z}. Hence we get:
σ^p = σ ∪ {{t},{z},{r,t},{r,z},{k,r,t},{k,r,z},{r,s,t},{r,s,z}, {k,r,s,t},{k,r,s,z},{k,r,t,z},{r,s,t,z}}.
Also
σ^β = σ ∪
{{t},{z},{k,r},{k,z},{r,s},{r,t},{r,z},{s,t},{s,z},{t,z},{k,r,s},{k,r,t},{k,r,z},{k,s,t},{k,s,z},{k,t,z},{r,s,t},{r,s,z},{r,t,z},{s,t,z}, {k,r,s,t},{k,r,s,z},{k,r,t,z},{k,s,t,z},{r,s,t,z}}.
Lemma 1 ([7]**)**
1. β-open sets are closed under arbitrary union.
2. **β**-closed sets are closed under arbitrary intersection.
Theorem 1 ([7]**)** If G is open and K is β-open, then G ∩ K is β-open.

IV. Developed operations on β-open subsets

In this article, we insert new operations on β-open subsets which are β- accumulation points, β-interior points,β-closure, β-derived, β-interior, β-border, β-frontier and β-exterior of subsets. Then, we study their topological properties.

Definition 1 ([8]) Let K be a subset of (M, σ). Then, β-closure of K in M is the intersection of all β-closed subsets of M which contains K and is defending by $Cl_\beta(K)$.

Remark 1 *The subset K is β-closed [8] if and only if K = Clβ(K).*

Example 2 *Let* $\sigma = \{M, \phi, \{r\}, \{s\}, \{r,s\}\}$ *is a topology on* $M = \{k,r,s\}$ *and* $K = \{k,s\}$ *be subset of M. Then, we get:*

$Cl(K) = \{k,s\}.Clp(K) = \phi$.[2] $\quad Cl\beta(K) = \{k,s\}$.

Theorem 2 *If K is a subset of a space M and* $m \in M$*, then the next sentences are equivalent:*

1. $(\forall H \in \tau\beta)\ (m \in H \Rightarrow K \cap H \neq \phi)$.
2. $m \in Cl\beta(K)$.

Proof 1 *(1) ⇒ (2). If* $m \notin Cl_\beta(K)$*, so we find β-closed set F where* $K \subseteq F$ *and* $m \notin F$*. Thus* $M \setminus F$ *is β-open set which contains m then* $K \cap (M \setminus F) \subseteq K \cap (M \setminus K) = \phi$ *which is a conflict. Therefore (2) is satisfied.*

(2) ⇒ (1). It's clear.

Proposition 1 *If K and R are two subsets of (M,σ), then the following statements hold:*

1. $K \subseteq Cl_\beta(K)$.
2. *If* $K \subseteq R$*, then* $Cl_\beta(K) \subseteq Cl_\beta(R)$.
3. $Cl_\beta(Cl_\beta(K)) = Cl_\beta(K)$.
4. *If* $Cl_\beta(K) \cap Cl_\beta(R) = \phi$*, then* $K \cap R = \phi$.
5. $Cl_\beta(K) \cup Cl_\beta(R) \subseteq Cl_\beta(K \cup R)$, $Cl_\beta(K \cap R) \subseteq Cl_\beta(K) \cap Cl_\beta(R)$.

Proof 2 *1. It follows from Definition 1.*

2. *Let* $K \subseteq R$ *and suppose that* $m \in Cl_\beta(K)$*. So by Theorem 2, for any β-open set H which contains m, we have* $K \cap H \neq \phi$*. But we know that* $K \subseteq R$ *. Then* $R \cap H \neq \phi$ *such that R is any β-open set which contains m. Thus,* $m \in Cl_\beta(R)$ *, then* $Cl_\beta(K) \subseteq Cl_\beta(R)$.
3. *We know that* $Cl_\beta(K)$ *is β-closed set, then* $Cl_\beta(Cl_\beta(K)) = Cl_\beta(K)$.
4. *Suppose that* $K \cap R \neq \phi$*, then there is* $m \in K \cap R$ *which implies* $m \in K$ *and* $m \in R$*. So by Part (1),* $m \in Cl_\beta(K)$ *and* $m \in Cl_\beta(R)$ *then* $Cl_\beta(K) \cap Cl_\beta(R) \neq \phi$.
5. *It's clear.*

The following examples prove that the inverse of parts (2) and the opposite inclusions of (5) is not true generally.

Example 3 *1. Let* $M = \{k,r,s\}$ *with the topology* $\sigma = \{\phi, M, \{r\}\}$*. For two subsets* $K = \{k\}$, $R = \{r\}$ *of M, So* $\{k\} = Cl_\beta(K) \subseteq Cl_\beta(R) = M$ *but* $K \nsubseteq R$*. Furthermore,* $K \cap R = \phi$ *but* $Cl_\beta(K) \cap Cl_\beta(R) = \{k\} \neq \phi$.

This proves that the inverse of Proposition 1 (2) is not true.

2. *Let* $M = \{k,r,s\}$ *with the topology* $\sigma = \{\phi, M, \{k\}, \{r\}, \{k,r\}\}$*. For two subsets* $K = \{k\}$ *and* $R = \{r\}$ *of M, so* $K \cup R = \{k,r\}$ *which implies,* $Cl_\beta(K) = \{k\}$, $Cl_\beta(R) = \{r\}$ *and hence,* $Cl_\beta(K \cup R) = M \nsubseteq \{k,r\} = Cl_\beta(K) \cup Cl_\beta(R)$*. This proves that the opposite inclusion of Proposition 1 (5) is not true.*
3. *In Part 1, since* $K \cap R = \phi$ *and hence,* $Cl_\beta(K \cap R) = \phi$*. But* $Cl_\beta(K) = \{k\}$ *and* $Cl_\beta(R) = M$ *, then* $Cl_\beta(K) \cap Cl_\beta(R) = \{k\} \nsubseteq$

$Cl_\beta(K \cap R) = \phi$*. This prove that the opposite inclusion of Proposition 1 (5) is not true.*

Definition 2 *For a subset K of a space (M,σ). An element* $m \in M$ *is called β-accumulation point of K if it satisfies the next condition:*

$\forall H \in \sigma^\beta, (m \in H \Rightarrow H \cap K \setminus \{m\} \neq \phi)$.

The family of all β-limit points of K is known as β-derived set of K and is defend by $D_\beta(K)$.

Also we have that, a point m in a space M is not β-accumulation point of $K \subseteq M$ *if and only if we find β-open set H in M where:*

$m \in H$ *and* $H \cap K \setminus \{m\} = \phi$.

or, equivalently,

$m \in H$ *and* $H \cap K = \phi$ *or* $H \cap K = \{m\}$.

or, equivalently,

$m \in H$ *and* $H \cap K \subseteq \{m\}$.

Example 4 *Assume the topology σ on $M = \{k,r,s,t,z\}$ given in Example 1. If $K = \{s,t,z\}$ is a subset of M, we get:*
$D(K) = \{t,z\}$.$D_p(K) = \{k,s\}$.[2] $D_\beta(K) = \phi$.

Theorem 3 *Let σ be a topology on a set M consists of ϕ , M , and $\{k\}$ for a constant $k \in M$, then $\sigma^p = \sigma^\beta$.*

Example 5 *If $M = \{k,r,s\}$ with $\sigma = \{M,\phi,\{k\}\}$. Hence we have the following:*

1. $\sigma^\beta = \{M,\phi,\{k\},\{k,r\},\{k,s\}\} = \sigma^p$.
2. If $K = \{k,s\}$, then $D(K) = \phi$ and $D_\beta(K) = D_p(K) = \{r,s\}$.
3. If $R = \{k\}$ and $S = \{r,s\}$, then $D_\beta(R) = \{r,s\}$, $D_\beta(S) = \phi$ and $D_\beta(R \cap S) = \{r,s\}$.

Theorem 4 *If σ_1 and σ_2 are topologies on M where $\sigma_1^\beta \subseteq \sigma_2^\beta$. Let K be any subset of M, then for all β-accumulation point of K with relative to σ_2 is β-accumulation point of K with relative to σ_1.*

Example 6 *Let $\sigma_1 = \{M,\phi,\{r\}\}$ and $\sigma_2 = \{M,\phi,\{r\},\{s,t\},\{r,s,t\}\}$ be topologies on a set $M = \{k,r,s,t\}$. Then $\sigma_1^\beta = \sigma_1 \cup \{\{k,r\},\{r,s\},\{r,t\},\{k,r,s\}$, $\{k,r,t\},\{r,s,t\}\}$ and $\sigma_2^\beta = \sigma_2 \cup \{\{s\},\{t\},\{k,r\},\{k,s\},\{k,t\},\{r,s\},\{r,t\},\{s,t\}$, $\{k,r,s\},\{k,r,t\},\{k,s,t\},\{r,s,t\}\}$. Note that $\sigma_1^\beta \subseteq \sigma_2^\beta$ and s is βaccumulation point of $K = \{r,s\}$ with relative to σ_1, but it is not βaccumulation point of K with relative to σ_2.*

Proposition 2 *If K and R are two subsets of (M,σ), then the following statements hold:*

1. $D_\beta(K) \subseteq D_p(K)$.
2. If $K \subseteq R$, then $D_\beta(K) \subseteq D_\beta(R)$.
3. $D_\beta(K) \cup D_\beta(R) \subseteq D_\beta(K \cup R)$, $D_\beta(K \cap R) \subseteq D_\beta(K) \cap D_\beta(R)$.
4. $D_\beta(D_\beta(K)) \setminus K \subseteq D_\beta(K)$.
5. $D_\beta(K \cup D_\beta(K)) \subseteq K \cap D_\beta(K)$.

Proof 3

1. *It is enough to know that every pre-open is β-open set.*
2. *Clear.*
3. *It's clear by (2).*
4. *Suppose that $m \in D_\beta(D_\beta(K)) \setminus K$ and $H \in \sigma^\beta$ such that $m \in H$. So $H \cap (D_\beta(K)\setminus\{m\})$ 6= ϕ. Suppose that $n \in H \cap (D_\beta(K)\setminus\{m\})$ which implies $n \in H$ and $n \in D_\beta(K)$, then $H \cap (K \setminus \{n\})$ =6 ϕ. If we choose $p \in H \cap (K \setminus \{n\})$, note that m 6= p and m /∈ K. So $(H \cap K) \setminus \{m\}$ 6= ϕ. Hence $m \in D_\beta(K)$.*
5. *Suppose that $m \in D_\beta(K \cup D_\beta(K))$. If $m \in K$, the result is clear. If m /∈ K. Thus*

$H \cap ((K \cup D_\beta(K))\setminus\{m\})$ 6= ϕ for every $H \in \sigma^\beta$ such that $m \in H$. Therefore $(H \cap K)\setminus\{m\}$ 6= ϕ or $H \cap (D_\beta(K)\setminus\{m\})$ 6= ϕ. From the first result we have $m \in D_\beta(K)$. If $H \cap (D_\beta(K) \setminus \{m\})$ 6= ϕ, then $m \in D_\beta(D_\beta(K))$. We have that m /∈ K and from (4) we get $m \in D_\beta(D_\beta(K)) \setminus K \subseteq D_\beta(K)$. Hence (5) is satisfied.

Generally, in Proposition 2, the inverse of Parts (1), (4) and (5), and the opposite inclusion of (2) need not be satisfied, and the evenness in Parts (3) does not satisfy as we see by the next example.

Example 7 *1. In Example 4. $D_p(K) = \{k,s\}$ and $D_\beta(K) = \phi$. This proves that the inverse of Proposition 2 (1) is not valid.*

2. *If $M = \{k,r,s,t\}$ with a topology $\sigma = \{M,\phi,\{k\},\{r\},\{k,r\},\{k,t\},\{r,s\}, \{k,r,s\}, \{k,r,t\}\} = \sigma^\beta$. Let $K = \{r,t\}$ and $R = \{k,r,s\}$ are two subsets of M, we obtain*

$D_\beta(K) = \{s\} \subseteq \{s,t\} = D_\beta(R)$,
but $K * R$. This proves that the opposite inclusion of Proposition 2 (2) is not true.
$\sigma^\beta = \{M,\phi,\{s\},\{t\},\{k,s\},\{k,t\},\{r,s\},\{r,t\},\{s,t\}, \{k,r,s\},\{k,r,t\},\{k,s,t\},\{r,s,t\}\}$.
If $K = \{r,s\}$ and $R = \{r,t\}$ are two subsets of M. So $D_\beta(K) = \phi = D_\beta(R)$, and hence
$D_\beta(K) \cup D_\beta(R) = \phi \subset \{k,r\} = D_\beta(K \cup R)$. Therefore the evenness in Proposition 2 (3) is not satisfied.

3. *In Example 5. $D_\beta(K) = \{r,s\} = D_\beta(R)$.Hence $D_\beta(K \cap R) = \phi \subseteq$*

$D_\beta(K) \cap D_\beta(R)$. Then the evenness in Proposition 2 (3) is not true.

4. *For a subset $K = \{r,s,t\}$ of M in above part (2), we have*

$D_\beta(D_\beta(K)) = D_\beta(\{k,r\}) = \phi$,
$D_\beta(D_\beta(K)) \setminus K = \phi \subseteq D_\beta(K) = \{k,r\}$,
and hence the evenness in Proposition 2 (4) is not satisfied.

5. *If* $R = \{s,t\}$ *is a subset of M in part (2), we obtain* $D_\beta(R) = \{k,r\}$, $R \cup D_\beta(R) = M$ *and* $D_\beta(M) = \{k,t\} \subseteq M$. *This proves that* $D_\beta(R \cup D_\beta(R)) 6= R \cup D_\beta(R) = M$. *Therefore the evenness in*

Proposition 2 (5) is not true.

Corollary 4.1 *If K is any subset of a space M, we have* $D_\beta(K) \subseteq Cl_\beta(K)$.
Theorem 5 *If K is any subset of a space M, then* $Cl_\beta(K) = K \cup D_\beta(K)$.
Theorem 6 *If M is a discrete space and K is a subset of M, then* $D_\beta(K) = \phi$.
Theorem 7 *If K is any subset of a space M and K is β-closed if and only if* $D_\beta(K) \subseteq K$.
Theorem 8 *If K is a subset of a space M and an element* $m \in M$ *is β-accumulation point of K, then m is β-accumulation point of* $K \setminus \{m\}$ *, too.*

Definition 3 ([8]) *If K is a subset of a space M. An element* $m \in M$ *is said to be β-interior point of K if we find β-open set H containing m where* $H \subseteq K$. *The family of all β-interior points of K is known as β-interior of K and is defend by* $I\, nt_\beta(K)$.

Example 8 *Taking into account the topology which is described in Example1.If* $K = \{k,r,s\}$ *is a subset of M:*
$I\, nt(K) = \{r\}$.$I\, nt_p(K) = \{r\}$.[2] $I\, nt_\beta(K) = \{k,r,s\}$.

Theorem 9 *If K is a subset of a space M. Then every pre-interior point of K is β-interior point of K (i.e.,* $I\, ntp(K) \subseteq I\, nt\beta(K)$ *).*

Proof 4 *Suppose that m is a pre-interior point of K, so we find a pre-open set H which contains m where* $H \subseteq K$. *We know that every pre-open set is β-open, then m is β-interior point of K.*

The next example proves that we find β-interior point of K which is not a pre-interior point of K.

Example 9 *In Example 8,* $I\, ntp(K) = \{r\}$ *and* $I\, nt\beta(K) = \{k,r,s\}$. *So k and s are β-interior points of K. But they are not pre-interior points of K.*

Proposition 3 *If K and R are two subsets of (M,σ), then the following statements hold:*

1. $I\, nt_\beta(K)$ *is the union for every β-open subsets of K.*
2. *K is β-open if and only if* $K = I\, nt_\beta(K)$.
3. $I\, nt_\beta(I\, nt_\beta(K)) = I\, nt_\beta(K)$.
4. $I\, nt_\beta(K) = K \setminus D_\beta(M \setminus K)$.
5. $M \setminus Cl_\beta(K) = I\, nt_\beta(M \setminus K)$.
6. $K \subseteq R \Rightarrow I\, nt_\beta(K) \subseteq I\, nt_\beta(R)$.
7. $I\, nt_\beta(K) \cup I\, nt_\beta(R) \subseteq I\, nt_\beta(K \cup R)$, $I\, nt_\beta(K \cap R) \subseteq I\, nt_\beta(K) \cap I\, nt_\beta(R)$.

Proof 5 *1. Suppose* $\{H_i \mid i \in \Lambda\}$ *be a class for every β-open subsets of K. If* $m \in I\, nt_\beta(K)$, *so we find* $j \in \Lambda$ *where* $m \in H_j \subseteq K$. *So* $m \in \cup_{i \in \Lambda} H_i$, *and hence* $I\, nt_\beta(K) \subseteq \cup_{i \in \Lambda} H_i$. *Conversely, if* $n \in \cup_{i \in \Lambda} H_i$, *so* $n \in H_l \subseteq K$ *for some* $l \in \Lambda$. *Then* $n \in I\, nt_\beta(K)$ *and* $\cup_{i \in \Lambda} H_i \subseteq I\, nt_\beta(K)$. *Hence,* $I\, nt_\beta(K) = \cup_{i \in \Lambda} H_i$.

2. *Clear.*
3. *It follows by considering Parts (1) and (2).*
4. *Suppose that* $m \in K \setminus D_\beta(M \setminus K)$, *hence* $m /\in D_\beta(M \setminus K)$ *and we find β-open set H which contains m where* $H \cap (M \setminus K) = \phi$. *Then* $m \in H \subseteq K$ *and so* $m \in Int_\beta(K)$. *This proves that* $K \setminus D_\beta(M \setminus K) \subseteq Int_\beta(K)$. *Since if* $m \in Int_\beta(K)$. *we know that* $Int_\beta(K) \in \sigma^\beta$. *Then* $Int_\beta(K) \cap (M \setminus K) = \phi$, *we get* $m /\in D_\beta(M \setminus K)$. *Hence* $Int_\beta(K) = K \setminus D_\beta(M \setminus K)$.
5. *Applying (4) also Theorem 5.*
6. *Clear.*
7. *It's by (6)*

Definition 4 *Let K be any subset of a space M, the subset*
$b_\beta(K) = K \setminus I\, nt_\beta(K)$
Is said to be β-border of K. Also the subset
$F\, r_\beta(K) = Cl_\beta(K) \setminus I\, nt_\beta(K)$
Is said to be β-frontier of K.
Also we have that, if *K* is *β*-closed subset of a space *M*, then $b_\beta(K) = F\, r_\beta(K)$.

Example 10 *1. Taking into account the topology given in Example 1. If K = {k,r,s} is a subset of M. Then*

$Int_p(K) = \{r\}$.[2] $Int_\beta(K) = \{k,r,s\}$. $b_p(K) = \{k,s\}$.[2]

$b_\beta(K) = \phi$.

Since K = {k,r,s} is β-closed, then

$Cl_p(K) = \{k,r,s\}$.[2] $Cl_\beta(K) = \{k,r,s\}$.

$Fr_p(K) = \{k,s\}$.[2] $Fr_\beta(K) = \phi$.

2. Consider the topological space (M,σ) which is described in Example 1. If K = {r,s,z} is a subset of M, we have $Int_\beta(K) = \{r,s,z\}$ *and* $Cl_\beta(K) = \{k,r,s,z\}$. *Therefore* $b_\beta(K) = \phi$ *and* $Fr_\beta(K) = \{k\}$.

Proposition 4 *Let K be a subset of a space M, then the following statements are valid:*

1. $b_\beta(K) \subseteq b_p(K)$.
2. $K = Int_\beta(K) \cup b_\beta(K)$, $Int_\beta(K) \cap b_\beta(K) = \phi$.
3. *K is β-open set if and only if* $b_\beta(K) = \phi$.
4. $Int_\beta(b_\beta(K)) = \phi$.
5. $b_\beta(b_\beta(K)) = b_\beta(K)$.
6. $b_\beta(K) = K \cap Cl_\beta(M \setminus K)$.
7. $b_\beta(K) = K \cap D_\beta(M \setminus K)$.

Proof 6 *1. We know that* $Int_p(K) \subseteq Int_\beta(K)$, *we have* $b_\beta(K) = K \setminus Int_\beta(K) \subseteq K \setminus Int_p(K) = b_p(K)$.

2. *Clear.*
3. *We know that* $Int_\beta(K)$ *is β-open, it follows from (3) that* $b_\beta(Int_\beta(K)) = \phi$.
4. *Suppose that* $m \in Int_\beta(b_\beta(K))$, *then* $m \in b_\beta(K) \subseteq K$ *and* $m \in Int_\beta(K)$. *Since* $Int_\beta(b_\beta(K)) \subseteq Int_\beta(K)$. *Then* $m \in b_\beta(K) \cap Int_\beta(K) = \phi$, *which is a conflict. Therefore* $Int_\beta(b_\beta(K)) = \phi$.
5. *Applying (4), we obtain:* $b_\beta(b_\beta(K)) = b_\beta(K) \setminus Int_\beta(b_\beta(K)) = b_\beta(K)$.
6. *Taking into account Proposition 3 (5), we get:* $b_\beta(K) = K \setminus Int_\beta(K) = K \setminus (M \setminus Cl_\beta(M \setminus K)) = K \cap Cl_\beta(M \setminus K)$.
7. *It's clear by using (6) also Theorem 5.*

Lemma 2 *If K is a subset of a space M and K is β-closed if and only if* $Fr_\beta(K) \subseteq K$.

Proposition 5 *Let K be a subset of a space M, then the following statements are valid:*

1. $Fr_\beta(K) \subseteq Fr_p(K)$.
2. $b_\beta(K) \subseteq Fr_\beta(K)$.
3. $Fr_\beta(K) = b_\beta(K) \cup (D_\beta(K) \setminus Int_\beta(K))$.
4. *K is β-open set if and only if* $Fr_\beta(K) = b_\beta(M \setminus K)$.
5. $Fr_\beta(K)$ *is β-closed.*
6. $Fr_\beta(Fr_\beta(K)) \subseteq Fr_\beta(K)$.

Proof 7*1. We know that* $Cl_p(K) \subseteq Cl_\beta(K)$ *and* $Int_p(K) \subseteq Int_\beta(K)$, *then*
$Fr_p(K) = Cl_p(K) \setminus Int_p(K) \subseteq Cl_\beta(K) \setminus Int_p(K) \supseteq Cl_\beta(K) \setminus Int_\beta(K) = Fr_\beta(K)$.

2. *We know that* $K \subseteq Cl_\beta(K)$, *we have* $b_\beta(K) = K \setminus Int_\beta(K) \subseteq Cl_\beta(K) \setminus Int_\beta(K) = Fr_\beta(K)$.
3. *Applying Theorem 5, we get:*
$Fr_\beta(K) = (K \cup D_\beta(K)) \cap (M \setminus Int_\beta(K)) = (K \setminus Int_\beta(K)) \cup (D_\beta(K) \setminus Int_\beta(K)) = b_\beta(K) \cup (D_\beta(K) \setminus Int_\beta(K))$.
4. *Suppose that K is β-open. Then*

$$
\begin{aligned}
Fr_\beta(K) &= b_\beta(K) \cup (D_\beta(K) \setminus Int_\beta(K)) \\
&= \phi \cup (D_\beta(K) \setminus K) \\
&= D_\beta(K) \setminus K \\
&= b_\beta(M \setminus K).
\end{aligned}
$$

by using (3), Proposition 4 (2), Proposition 3 (2) and Proposition 4 (7). opposite direction, assume that $Fr_\beta(K) = b_\beta(M \setminus K)$. *Hence*

$$
\begin{aligned}
\phi &= Fr_\beta(K) \setminus b_\beta(M \setminus K) \\
&= (Cl_\beta(K) \setminus Int_\beta(K)) \setminus ((M \setminus K) \setminus Int_\beta(M \setminus K)) = K \setminus Int_\beta(K).
\end{aligned}
$$

by Part (2) and (3) of Proposition 3, and hence $K \subseteq Int_\beta(K)$. *We know that* $Int_\beta(K) \subseteq K$, *then* $Int_\beta(K) = K$, *hence by Proposition 3 (2) that K is β-open.*

5. we have

$$Cl_\beta(Fr_\beta(K)) = Cl_\beta(Cl_\beta(K) \cap Cl_\beta(M \setminus K))$$

$\subseteq \quad Cl_\beta(Cl_\beta(K)) \cap Cl_\beta(Cl_\beta(M \setminus K))$

$= \quad Cl_\beta(K) \cap Cl_\beta(M \setminus K)$

$= \quad Fr_\beta(K).$

Clearly $Fr_\beta(K) \subseteq Cl_\beta(Fr_\beta(K))$, *and hence* $Fr_\beta(K) = Cl_\beta(Fr_\beta(K))$. *Therefore* $Fr_\beta(K)$ *is* β*-closed.*

6. it's by Part (5) also Lemma 2.

The opposite inclusions of Parts (1) and (2) of Proposition 5 are not satisfied generally as we see by the next example.

Example 11 *1. In Example 10(1), If* $K = \{k,r,z\}$. *Then* $F\ r_\beta(K) = \phi \subseteq \{r,z\} = F\ r_p(K)$. *This proves that the opposite inclusion of Proposition 5 (1) is not true.*

2. In Example 10 (2) this proves that the opposite inclusion of Proposition 5 (2) is not true generally.

Definition 5 *Let K be a subset of a space M, the subset:*

$Ext_\beta(K) = I\ nt_\beta(M \setminus K)$

Is called β*-exterior of K.*

Example 12 *Taking into account the topology given in Example 1. Let* $K = \{r,s,t\}$ *be a subset of M, we have :*

$Ext(K) = \phi . Ext_p(K) = \{z\}$.[2] $\qquad Ext_\beta(K) = \{k,z\}$.

Proposition 6 *If K and R are two subsets of* (M,σ), *then the following statements hold:*

1. $Ext_p(K) \subseteq Ext_\beta(K)$.
2. $Ext_\beta(K)$ *is* β*-open.*
3. $Ext_\beta(Ext_\beta(K)) = I\ nt_\beta(Cl_\beta(K)) \supseteq I\ nt_\beta(K)$.
4. *If* $K \subseteq R$ *then* $Ext_\beta(R) \subseteq Ext_\beta (K)$.
5. $Ext_\beta(K \cup R) \subseteq Ext_\beta(K) \cap Ext_\beta(R)$, $Ext_\beta(K \cap R) \supseteq Ext_\beta(K) \cup Ext_\beta(R)$.
6. $M = I\ nt_\beta(K) \cup Ext_\beta(K) \cup F\ r_\beta(K)$.

Proof 8 *1. Applying Theorem 9, we get:*

$Ext_p(K) = I\ nt_p(M \setminus K) \subset I\ nt_\beta(M \setminus K) = Ext_\beta(K)$.

2. *It's by Lemma 1 (1) and Proposition 3 (1).*
3. *Using Parts (4) also (6) of Proposition 3, we obtain:*

$Ext_\beta(Ext_\beta(K)) \quad = \quad Ext_\beta(I\ nt_\beta(M \setminus K))$

$= I\ nt_\beta(M \setminus I\ nt_\beta(M \setminus K))$

$= I\ nt_\beta(Cl_\beta(K)) \supset I\ nt_\beta(K)$.

4. *Clear.*
5. *Taking into account Proposition 3 (7), we obtain:*

$Ext_\beta(K \cup R) \quad = \quad I\ nt_\beta(M \setminus (K \cup R))$

$= I\ nt_\beta((M \setminus K) \cap (M \setminus R)) \subseteq I\ nt_\beta(M \setminus K) \cap I\ nt_\beta(M \setminus R)$

$= Ext_\beta (K) \cap Ext_\beta(R)$.

6. *Clear.*

The opposite inclusions of Parts (1), (4), (5) of Proposition 6 are not satisfied generally as we see by the next example.

Example 13 *Taking into account the topology given in Example 1.*

1. *If* $K = \{r,s,t\}$. *Then,* $Ext_p (K) = \{z\}$ *[2] and* $Ext_\beta (K) = \{k,z\}$. *This opposite inclusion of Proposition 6 (1) is not satisfied.*
2. *If* $K = \{t,z\}$ *and* $R = \{k,t,z\}$. *Then,* $Ext_\beta(R) = \{r,s\} \subseteq \{k,r,s\} = Ext_\beta(K)$. *This proves that the opposite inclusion of (4) in Proposition 6 is not true.*
3. *If* $K = \{r,s,t\}$ *and* $R = \{k,z\}$. *Then,* $Ext_\beta (K \cup R) = \phi$ 6= $\{k\} = \{k,z\} \cap \{k,r,s,t\} = Ext_\beta(K) \cap Ext_\beta(R)$. *This proves that the evenness in Proposition 6 (5) is not satisfied.*

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