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A New k - Step Iterative Scheme in Convex Metric Space

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Abstract: *In this paper, we introduce a general k- step implicit iterative scheme and we prove that the Mann, Ishikawa and Noor implicit iterative schemes are special cases of our result. Moreover, C programming is used to study and compare the rate of convergence with numerical examples. Finally, we deduce a new result that implicit iterative schemes converge faster as compared to explicit iterative schemes.*

I. Introduction

Recently, many papers have been developed on explicit iterations in various spaces[1,6], but only few works have considered implicit iterations (regarding convergence rate and data dependence)[7,15]. Implicit iterations are advantageous over explicit iterations for non linear problems as they provide better approximation of fixed points and can be used in many applications when explicit iterations are inefficient. Approximations of fixed points in computer oriented program using implicit iterations can reduce the cost of computation for fixed point problems. The study of stability of iterations enjoys a celebrated place in applied sciences and engineering especially the numerical methods derived to solve the engineering problems due to chaotic behaviour of functions in discrete dynamics and other numerical computations. Data dependence of fixed points is a related and new issue which has been studied by many authors[4,16]. In computational mathematics, it is of theoretical and practical importance to compare the convergence rate of iterations and to find out, if possible, which one of them converges more rapidly to the fixed point. Recent works in this direction are[1,3,4,17-19] .in concrete, a fixed point iteration is valuable from a numerical point of view and is useful for applications if it satisfies the following requirements:

(a) It converges to fixed point of an operator;
(b) It is T- stable;
(c) It is faster as compared to other iterations existing in the literature;
(d) It shows data dependence results.

Motivated by the fact that three step iterations give better approximation than the two step and two step iterations give better approximations then one step iteration[20], we define a new and more general k- step implicit iteration (Ik) which satisfies the above requirements. Now with a complexity and simplicity of the situation one can fix the number of steps. Consequently all the one step, two step and three step iterations can be derived as special cases of this iterative scheme.

Let K be a non empty convex subset of a convex metric metric space X and let $T: K \to K$ be a given mapping. For the real sequences $\left\{\alpha_n^{(1)}\right\}_{n=0}^{\infty}$, $\left\{\alpha_n^{(2)}\right\}_{n=0}^{\infty}$, $\left\{\alpha_n^{(3)}\right\}_{n=0}^{\infty}$ in $[0,1]$, Noor iteration[21] in convex metric spaces can be written as

$$\begin{aligned} x_n^{(1)} &= W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(2)}, \alpha_n^{(1)}\right), \\ x_{n-1}^{(2)} &= W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(3)}, \alpha_n^{(2)}\right), \qquad \text{(N)} \\ x_{n-1}^{(3)} &= W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(1)}, \alpha_n^{(3)}\right), \qquad n = 0,1,2,3,\dots \end{aligned}$$

Putting $\alpha_n^{(3)} = 1$ in (N) we get well known Ishikawa iteration[22,23] in convex metric spaces:

$$\begin{aligned} x_n^{(1)} &= W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(2)}, \alpha_n^{(1)}\right), \\ x_{n-1}^{(2)} &= W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(1)}, \alpha_n^{(2)}\right), \qquad n = 0,1,2,3,\dots \qquad \text{(I)} \end{aligned}$$

Putting $\alpha_n^{(3)} = \alpha_n^{(2)} = 1$ in (N), we get well known Mann iteration[23,24] in convex metric spaces:

$$x_n^{(1)} = W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(1)}, \alpha_n^{(1)}\right), \qquad n = 0,1,2,3,\dots \qquad \text{(M)}$$

For $x_0 \in K$, we define the following k- step implicit iterations:

$$\begin{aligned} x_n^{(1)} &= W\left(x_{n-1}^{(2)}, Tx_n^{(1)}, \alpha_n^{(1)}\right), \\ x_n^{(2)} &= W\left(x_{n-1}^{(3)}, Tx_n^{(2)}, \alpha_n^{(2)}\right), \\ x_n^{(3)} &= W\left(x_{n-1}^{(4)}, Tx_n^{(3)}, \alpha_n^{(3)}\right), \end{aligned}$$

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$x_n^{(k)} = W\left(x_{n-1}^{(1)}, Tx_n^{(k)}, \alpha_n^{(k)}\right),$ (Ik)

where $\left\{\alpha_n^{(1)}\right\}_{n=0}^{\infty}, \left\{\alpha_n^{(2)}\right\}_{n=0}^{\infty}, \left\{\alpha_n^{(3)}\right\}_{n=0}^{\infty}, \ldots, \left\{\alpha_n^{(k)}\right\}_{n=0}^{\infty}$ are sequences in [0,1].

Equivalent iterative equations in linear system can be written as:

$x_{n-1}^{(1)} = \alpha_n^{(1)} x_{n-1}^{(2)} + \left(1 - \alpha_n^{(1)}\right) Tx_n^{(1)},$

$x_{n-1}^{(2)} = \alpha_n^{(2)} x_{n-1}^{(3)} + \left(1 - \alpha_n^{(2)}\right) Tx_{n-1}^{(2)},$

$x_{n-1}^{(3)} = \alpha_n^{(3)} x_{n-1}^{(4)} + \left(1 - \alpha_n^{(3)}\right) Tx_{n-1}^{(3)},$

...

...

...

$x_{n-1}^{(k)} = \alpha_n^{(k)} x_{n-1}^{(1)} + \left(1 - \alpha_n^{(k)}\right) Tx_{n-1}^{(k)},$

Put $\alpha_n^{(k)} = \alpha_n^{(k-1)} = \alpha_n^{(k-2)} = \cdots = \alpha_n^{(4)} = 1$, we get well known implicit Noor iteration[7] in convex metric spaces is defined as follows:

$x_n^{(1)} = W\left(x_{n-1}^{(2)}, Tx_n^{(1)}, \alpha_n^{(1)}\right),$

$x_{n-1}^{(2)} = W\left(x_{n-1}^{(3)}, Tx_{n-1}^{(2)}, \alpha_n^{(2)}\right),$ (IN)

$x_{n-1}^{(3)} = W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(3)}, \alpha_n^{(3)}\right), \quad n = 0,1,2,3, \ldots$

Putting $\alpha_n^{(3)} = 1$ in (IN) we get well known implicit Ishikawa iteration[25] in convex metric spaces:

$x_n^{(1)} = W\left(x_{n-1}^{(2)}, Tx_n^{(1)}, \alpha_n^{(1)}\right),$

$x_{n-1}^{(2)} = W\left(x_{n-1}^{(1)}, Tx_{n-1}^{(2)}, \alpha_n^{(2)}\right), n = 0,1,2,3, \ldots$ (II)

Putting $\alpha_n^{(3)} = \alpha_n^{(2)} = 1$ in (IN), we get well known implicit Mann iteration[2,6,13,26] in convex metric spaces:

$x_n^{(1)} = W\left(x_{n-1}^{(1)}, Tx_n^{(1)}, \alpha_n^{(1)}\right), \quad n = 0,1,2,3, \ldots$ (IM)

In Zamfirescu[27], established a nice generalization of a Banach's fixed point theorem by employing the following contractive condition: For a mapping $T: E \to E$, there exist real number α, β, γ satisfying, $0 \le \alpha \le 1, 0 \le \beta \le \frac{1}{2}, 0 \le \gamma \le \frac{1}{2}$, respectively such that for each $x, y \in E$, at least one of the following is true:

(z_1) $d(Tx, Ty) \le \alpha d(x, y)$

(z_2) $d(Tx, Ty) \le \beta[d(x, Tx) + d(y, Ty)]$

(z_3) $d(Tx, Ty) \le \gamma[d(x, Ty) + d(y, Tx)]$.

The mapping $T: E \to E$ satisfying (2) is called the Zamfirescu contraction.

Z- operators are equivalent to the following contractive condition:

$d(Tx, Ty) \le c \max\left\{d(x, y), \frac{\{d(x,Tx)+d(y,Ty)\}}{2}, \frac{\{d(x,Ty)+d(y,Tx)\}}{2}\right\}$ (1)

$\forall x, y \in X, 0 < c < 1.$

The contractive condition (1) implies

$d(Tx, Ty) \le 2a d(x, Tx) + a d(x, y), \qquad \forall x, y \in X, \lambda \in R$ (2)

In[5], Rhoades used the following more general contractive condition than (2): there exist $c \in [0, 1)$ such that

$d(Tx, Ty) \le c \max\left\{d(x, y), \frac{\{d(x,Tx)+d(y,Ty)\}}{2}, d(x, Ty), d(y, Tx)\right\} \qquad \forall x, y \in X.$ (3)

In[28], Osilike used a more general contractive condition than those of Rhoades: there exist $a \in [0, 1), L \ge 0$ such that

$d(Tx, Ty) \le L d(x, Tx) + a d(x, y) \qquad \forall x, y \in X.$ (4)

We use the contractive condition due to Imoru and Olatinwo[29], which is more general than (4): there exist $a \in [0,1)$ and a monotone increasing function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$, such that

$d(Tx, Ty) \le \varphi\left(d(x, Tx)\right) + a d(x, y), \qquad a \in [0,1), \qquad \forall x, y \in X.$ (5)

Also, we use the following definitions and lemmas to achieve our main results.

Definition 1(see [25]): A map $W: X^2 \times [0,1] \to X$ is a convex structure on X if

$d\left(u, W(x, y, \lambda)\right) \le \lambda d(u, x) + (1 - \lambda) d(u, y)$ (6)

For all $x, y, u \in X$and $\lambda \in [0,1]$. A metric space (X, d) together with a convex structure W is known as convex metric space and denoted by (X, d, W). A nonempty set C of a convex metric space if $W(x, y, \lambda) \in C$ for all $x, y \in C$and $\lambda \in [0,1]$.

All normed spaces and their subsets are the examples of convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space(see [25,30]). After that several authors extended this concept in many ways: one such convex structure is hyperbolic space which was introduced by kohlenbach[31] as follows.

Definition 2(see [31]): A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W: X^2 \times [0,1] \to X$ satisfying
(W1) $d(z, W(x, y, \lambda)) \leq (1-\lambda)d(z, x) + \lambda d(z, y)$,
(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| d(x, y)$,
(W3) $W(x, y, \lambda) = W(y, x, 1-\lambda)$,
(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1-\lambda)d(x, y) + \lambda d(z, w)$
For all $x, y, z, w \in X$and $\lambda, \lambda_1, \lambda_2 \in [0,1]$.

Evidently every hyperbolic space is a convex metric space but the converse may not be true. For example, if we take $X = R$, $W(x, y, \lambda) = \lambda x + (1-\lambda)y$ and define $d(x, y) = |x-y|/(1+|x-y|)$ for $x, y \in \mathbb{R}$, then (X, d, W) is a convex metric space but not a hyperbolic space.
The stability of explicit as well as implicit iterations has extensively been studied by many authors[4,9,23,29,32-34] due to its increasing importance in computational mathematics, especially due to revolution in computer programming. The concept of T- stability in convex metric space setting was given by Olatinwo[23].

Definition 3(see [23]): Let (X, d, W) be a convex metric space and let $T: X \to X$ a self mapping.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative scheme involving T which is defined by
$x_{n+1} = f_{T,\alpha_n}^{x_n}$, $\quad n = 0,1,2,3 ...,$ (7)
Where $x_0 \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure such that $\alpha_n \in [0,1]$. Suppose that $\{x_n\}$ converges to a fixed point $p$of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\varepsilon_n = d(y_{n+1}, f_{T,\alpha_n}^{y_n})$. Then the iteration (7) is said to be T- stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$implies $\lim_{n\to\infty} y_n = p$.

Lemma 4(see [4, 17]): If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying
$u_{n+1} \leq \delta u_n + \epsilon_n$, $\quad n = 0,1,2,3, ...,$ (8)
One has $\lim_{n\to\infty} u_n = 0$.

Definition 5(see [17]): Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two real convergent sequences with limits $a$and b, respectively. Then $\{\alpha_n\}$ is said to converge faster than $\{\beta_n\}$ if
$\lim_{n\to\infty} \left|\frac{\alpha_n - a}{\beta_n - b}\right| = 0$
(9)

Definition 6(see [17]): Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iterations that converge to the same fixed point p on a normed space X such that the error estimates
$\|u_n - p\| \leq a_n$,
$\|v_n - p\| \leq b_n$. (10)
Are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive integers of positive numbers (converging to zero). If $\{a_n\}$ converge faster $\{b_n\}$, then one says that $\{u_n\}$ converge faster to p than $\{v_n\}$.

Definition 7(see [16]): Let T, T_1 be two operators on X. One says T_1 is approximate of T if, for all $x \in X$ and for a fixed $\varepsilon > 0$, one has $d(Tx, T_1x) \leq \varepsilon$.

Lemma 8(see [4, 16]): Let $\{a_n\}_{n=0}^{\infty}$ be a non negative sequence for which there exists $n_0 \in N$ such that, for all $n \geq n_0$, one has the following inequality:
$a_{n+1} \leq (1 - r_n)a_n + r_n t_n$, (11)
where $r_n \in (0,1)$, for all$n \in N$, $\sum_{n=1}^{\infty} r_n = \infty$, and $t_n \geq 0 \forall n \in N$.
Then, $0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup t_n$.
Having introduced the implicit general k- step iteration (Ik), we use it to prove the results concerning convergence, stability, and convergence rate for contractive condition (7) in convex metric spaces. Also, data dependence result of the same iteration is proved in hyperbolic spaces.

II. Main Results

2.1) Convergence and stability results of new implicit iteration in convex metric spaces

Theorem 9. Let K be a nonempty closed convex subset of a convex metric space X and let T be a quasi contractive operator satisfying (5) with $F(T) \neq \emptyset$. Then, for $x_0 \in C$, then sequence $\{x_n\}$ defined by (Ik), with $\sum(1-\alpha_n) = \infty$, converges to the fixed point of T.

Proof: using (IN) and (5), we have, for $p = F(T)$,

$$d(x_n^{(1)}, p) = d(W(x_{n-1}^{(2)}, Tx_n^{(1)}, \alpha_n^{(1)}), p) \leq \alpha_n^{(1)} d(x_{n-1}^{(2)}, p) + (1-\alpha_n^{(1)}) d(Tx_n^{(1)}, p)$$
$$\leq \alpha_n^{(1)} d(x_{n-1}^{(2)}, p) + (1-\alpha_n^{(1)}) a d(x_n^{(1)}, p), \tag{12}$$

which further implies

$$\{1-(1-\alpha_n^{(1)})\} d(x_n^{(1)}, p) \leq \alpha_n^{(1)} d(x_{n-1}^{(2)}, p)$$
$$d(x_n^{(1)}, p) \leq \frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a} d(x_{n-1}^{(2)}, p). \tag{13a}$$

Again from (Ik), we have the following estimates:

$$d(x_{n-1}^{(2)}, p) = d(W(x_{n-1}^{(3)}, Tx_{n-1}^{(2)}, \alpha_n^{(2)}), p) \leq \alpha_n^{(2)} d(x_{n-1}^{(3)}, p) + (1-\alpha_n^{(2)}) d(Tx_{n-1}^{(2)}, p)$$
$$\leq \alpha_n^{(2)} d(x_{n-1}^{(3)}, p) + (1-\alpha_n^{(2)}) a d(x_{n-1}^{(2)}, p),$$

which gives

$$d(x_{n-1}^{(2)}, p) \leq \frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a} d(x_{n-1}^{(3)}, p), \tag{13b}$$

Similarly,

$$d(x_{n-1}^{(3)}, p) \leq \frac{\alpha_n^{(3)}}{1-(1-\alpha_n^{(3)})a} d(x_{n-1}^{(4)}, p), \tag{13c}$$

and so on. Such that final equation is

$$d(x_{n-1}^{(k)}, p) \leq \frac{\alpha_n^{(k)}}{1-(1-\alpha_n^{(k)})a} d(x_{n-1}^{(1)}, p). \tag{13d}$$

Using above equations we have,

$$d(x_n^{(1)}, p) \leq \left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left(\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right)\left(\frac{\alpha_n^{(3)}}{1-(1-\alpha_n^{(3)})a}\right)\cdots\left(\frac{\alpha_n^{(k)}}{1-(1-\alpha_n^{(k)})a}\right) d(x_{n-1}^{(1)}, p). \tag{14}$$

If we take $\alpha_n^{(1)}/[1-(1-\alpha_n^{(1)})a] = A_n/B_n$, then

$$1-\frac{A_n}{B_n} = 1-\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a} = \frac{1-[(1-\alpha_n^{(1)})a+\alpha_n^{(1)}]}{1-(1-\alpha_n^{(1)})} \geq 1-[(1-\alpha_n^{(1)})a+\alpha_n^{(1)}]. \tag{15}$$

And hence

$$\frac{A_n}{B_n} \leq (1-\alpha_n^{(1)})a + \alpha_n^{(1)} = 1-(1-a)(1-\alpha_n^{(1)}). \tag{16a}$$

Similarly, with ease we can prove that

$$\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a} \leq \alpha_n^{(2)} + (1-\alpha_n^{(2)})a = 1-(1-\alpha_n^{(2)})(1-a) \leq 1, \tag{16b}$$

$$\frac{\alpha_n^{(3)}}{1-(1-\alpha_n^{(3)})} \leq \alpha_n^{(3)} + (1-\alpha_n^{(3)})a = 1-(1-\alpha_n^{(3)})(1-a) \leq 1, \tag{16c}$$

And so on such that

$$\frac{\alpha_n^{(k)}}{1-(1-\alpha_n^{(k)})a} \leq \alpha_n^{(k)} + (1-\alpha_n^{(k)})a = 1-(1-\alpha_n^{(k)})(1-a) \leq 1. \tag{16d}$$

Using these equations we get,

$$d(x_n^{(1)}, p) \leq [1-(1-\alpha_n^{(1)})(1-a)] d(x_{n-1}^{(1)}, p)$$
$$\leq \prod_{i=1}^{n} [1-(1-\alpha_i^{(1)})(1-a)] d(x_0, p) \leq e^{-\sum_{i=1}^{n}(1-\alpha_i^{(1)})(1-a)} d(x_0, p). \tag{17}$$

But $\sum(1-\alpha_i)=\infty$; hence () yields $\lim_{n\to\infty} d\left(x_n^{(1)},p\right)=0$. Therefore $\{x_n\}$ converges to p.

Theorem 10. Let K be a nonempty closed convex subset of a convex metric space X and let T be a quasi-contractive operator satisfying (5) with $F(T)=\emptyset$. Then, for $x_0\in C$, the sequence $\{x_n\}$ defined by (Ik) with $\alpha_n^{(1)}<\alpha<1, \sum\left(1-\alpha_n^{(1)}\right)=\infty$, is T- stable.

Proof: Suppose that $\left\{p_n^{(1)}\right\}_{n=0}^{\infty}\subset K$ is an arbitrary sequence, $\varepsilon_n=d\left(p_n^{(1)},W\left(p_{n-1}^{(2)},Tp_n^{(1)},\alpha_n^{(1)}\right)\right)$, where $p_{n-1}^{(2)}=W\left(p_{n-1}^{(3)},Tp_{n-1}^{(2)},\alpha_n^{(2)}\right)$, $p_{n-1}^{(3)}=W\left(p_{n-1}^{(4)},Tp_{n-1}^{(3)},\alpha_n^{(3)}\right)$, and so on such that $p_{n-1}^{(k)}=W\left(p_{n-1}^{(1)},Tp_{n-1}^{(k)},\alpha_n^{(k)}\right)$ and let $\lim_{n\to\infty}\varepsilon_n=0$.

Then, using (5) we have

$$\begin{aligned} d\left(p_n^{(1)},p\right) &\le d\left(p_n^{(1)},W\left(p_{n-1}^{(2)},Tp_n^{(1)},\alpha_n^{(1)}\right)\right)+d\left(W\left(p_{n-1}^{(2)},Tp_n^{(1)},\alpha_n^{(1)}\right)\right)\\ &\le \varepsilon_n+\alpha_n^{(1)}d\left(p_{n-1}^{(2)},p\right)+\left(1-\alpha_n^{(1)}\right)d\left(Tp_n^{(1)},p\right)\\ &\le \varepsilon_n+\alpha_n^{(1)}d\left(p_{n-1}^{(2)},p\right)+\left(1-\alpha_n^{(1)}\right)\varphi d\left(p_n^{(1)},p\right)+\left(1-\alpha_n^{(1)}\right)ad\left(p_n^{(1)},p\right), \end{aligned} \tag{18}$$

which implies

$$\left\{1-\left(1-\alpha_n^{(1)}\right)a\right\}d\left(p_n^{(1)},p\right)\le\varepsilon_n+\alpha_n^{(1)}d\left(p_{n-1}^{(2)},p\right), \tag{19}$$

and therefore,

$$d\left(p_n^{(1)},p\right)\le\frac{\alpha_n^{(1)}}{1-\left(1-\alpha_n^{(1)}\right)a}d\left(p_{n-1}^{(2)},p\right)+\frac{\varepsilon_n}{1-\left(1-\alpha_n^{(1)}\right)a}. \tag{20}$$

Using equations(16a), (20) becomes

$$d\left(p_n^{(1)},p\right)\le\left[1-\left(1-\alpha_n^{(1)}\right)(1-a)\right]d\left(p_{n-1}^{(2)},p\right)+\frac{\varepsilon_n}{1-\left(1-\alpha_n^{(1)}\right)a}. \tag{21a}$$

Now, using equations (16b), (16c) and (16d) we have the following estimates:

$$d\left(p_{n-1}^{(2)},p\right)\le\frac{\alpha_n^{(2)}}{1-\left(1-\alpha_n^{(2)}\right)}d\left(p_{n-1}^{(3)},p\right)\le d\left(p_{n-1}^{(3)},p\right), \tag{21b}$$

$$d\left(p_{n-1}^{(3)},p\right)\le\frac{\alpha_n^{(3)}}{1-\left(1-\alpha_n^{(3)}\right)a}d\left(p_{n-1}^{(4)},p\right)\le d\left(p_{n-1}^{(4)},p\right), \tag{21c}$$

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$$d\left(p_{n-1}^{(k)},p\right)\le\frac{\alpha_n^{(k)}}{1-\left(1-\alpha_n^{(k)}\right)a}d\left(p_{n-1}^{(1)},p\right)\le d\left(p_{n-1}^{(1)},p\right). \tag{21d}$$

Using $\alpha_n^{(1)}\le\alpha<1$and $a\in[0,1)$, we have $1-\left(1-\alpha_n^{(1)}\right)(1-a)<1$. Hence using lemma 4, together with estimates in equations (21a), (21b), (21c) and (21d)yields $\lim_{n\to\infty}p_n^{(1)}=p$.

Conversely if we let $\lim_{n\to\infty}p_n^{(1)}=p$, then using contractive condition (5), it is easy to see that $\lim_{n\to\infty}\varepsilon_n=0$. Therefore, the iteration (Ik) is T- stable.

Remark 11: As contractive condition (5) is more general than those of (1)-(4) the convergence and stability results for implicit k-step iterations (Ik) using contractive conditions (1)-(4) can be obtained as special cases

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Remark 12: As implicit Mann iterations (IM) and implicit Ishikawa iterations (II), implicit Noor iterations (IN) are of special cases of implicit k-step iteration (Ik), results are similar to theorems 9 and 10 hold for these iterations as well.

2.2) Convergence rate of implicit iterations

Theorem 13: Let K be a nonempty convex subset of a convex metric space X and let T be a quasi- contractive operator satisfying (5) with $F(T)\neq\emptyset$. Then for$x_0\in C$, the sequence $\{x_n\}$ defined by (Ik) with $\sum\left(1-\alpha_n^{(1)}\right)=\infty$, $\alpha_n^{(1)}\le\alpha<1$, converges faster than implicit mann (IM), implicit Ishikawa(II) and implicit Noor iterations to the fixed point of T. Moreover, implicit iterations converge faster than the corresponding explicit iterations.

Proof: For implicit Mann iterations (IM), we have

$$\begin{aligned} d\left(x_n^{(1)},p\right)=d\left(W\left(x_{n-1}^{(1)},Tx_n^{(1)},\alpha_n^{(1)}\right),p\right) &\le \alpha_n^{(1)}d\left(x_{n-1}^{(1)},p\right)+\left(1-\alpha_n^{(1)}\right)d\left(Tx_n^{(1)},p\right)\\ &\le \alpha_n^{(1)}d\left(x_{n-1}^{(1)},p\right)+\left(1-\alpha_n^{(1)}\right)ad\left(x_n^{(1)},p\right), \end{aligned} \tag{22}$$

which further yield,

$[1-(1-\alpha_n^{(1)})a]d(x_n^{(1)},p) \leq \alpha_n^{(1)} d(x_{n-1}^{(1)},p),$ (23)

and so

$d(x_n^{(1)},p) \leq \frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a} d(x_{n-1}^{(1)})p.$ (24a)

Similarly for implicit Ishikawa (II) and implicit Noor (IN) iterations, we have the estimates (33) and (34), respectively, as follows:

$d(x_n^{(1)},p) \leq \left[\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right]\left[\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right] d(x_{n-1}^{(1)},p),$ (24b)

$d(x_n^{(1)},p) \leq \left[\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right]\left[\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right]\left[\frac{\alpha_n^{(3)}}{1-(1-\alpha_n^{(3)})a}\right] d(x_{n-1}^{(1)},p),$ (24c)

And for implicit k- step iteration the estimate is:

$d(x_n^{(1)},p) \leq \left[\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right]\left[\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right] \dots\dots\dots \left[\frac{\alpha_n^{(k)}}{1-(1-\alpha_n^{(1)})a}\right] d(x_{n-1}^{(1)},p).$ (24d)

Also for explicit mann iteration (M), we have

$$\begin{aligned} d(x_n^{(1)},p) = d(W(x_{n-1}^{(1)},Tx_{n-1}^{(1)},\alpha_n^{(1)}),p) &\leq \alpha_n^{(1)} d(x_{n-1}^{(1)},p) + (1-\alpha_n^{(1)}) d(Tx_{n-1}^{(1)},p) \\ &\leq \alpha_n^{(1)} d(x_{n-1}^{(1)},p) + (1-\alpha_n^{(1)}) a d(x_{n-1}^{(1)},p) \leq [\alpha_n^{(1)} + (1-\alpha_n^{(1)})a] d(x_{n-1}^{(1)},p). \end{aligned} \tag{25}$$

For explicit Ishikawa iteration(I), we have

$$\begin{aligned} d(x_n^{(1)},p) = d(W(x_{n-1}^{(1)},Tx_{n-1}^{(1)},\alpha_n^{(1)}),p) &\leq \alpha_n^{(1)} d(x_{n-1}^{(1)},p) + (1-\alpha_n^{(1)}) d(Tx_{n-1}^{(1)},p) \\ &\leq \alpha_n^{(1)} d(x_{n-1}^{(1)},p) + (1-\alpha_n^{(1)}) a d(x_{n-1}^{(1)},p) \leq [\alpha_n^{(1)} + (1-\alpha_n^{(1)})a] d(x_{n-1}^{(1)},p) \end{aligned}$$

(26)

For explicit Ishikawa iteration (I), we have,

$$\begin{aligned} d(x_n^1,p) = d(W(x_{n-1}^1,Tx_{n-1}^2,\alpha_n^1),p) &\leq \alpha_n^1 d(x_{n-1}^1,p) + (1-\alpha_n^1) d(Tx_{n-1}^2,p) \\ &\leq \alpha_n^{(1)} d(x_{n-1}^{(1)},p) + (1-\alpha_n^{(1)}) a d(x_{n-1}^{(2)},p), \end{aligned} \tag{27a}$$

$d(x_{n-1}^{(2)},p) = d(W(x_{n-1}^{(1)},Tx_{n-1}^{(2)},\alpha_n^{(2)}),p) \leq [\alpha_n^{(2)} + (1-\alpha_n^{(2)})a] d(x_{n-1}^{(1)},p).$(27b)

Using (27b), (27a) becomes,

$d(x_n^{(1)},p) \leq [\alpha_n^{(1)} + (1-\alpha_n^{(1)})a(\alpha_n^{(2)} + (1-\alpha_n^{(2)})a)] d(x_{n-1}^{(1)},p).$ (28)

Similarly, for explicit Noor Iteration(N), we have,

$$d(x_n^{(1)},p) \leq \left\{\alpha_n^{(1)} + (1-\alpha_n^{(1)}) a \alpha_n^{(2)} + (1-\alpha_n^{(1)})(1-\alpha_n^{(2)}) a^2 [\alpha_n^{(3)} + (1-\alpha_n^{(3)})a]\right\} d(x_{n-1}^{(1)},p). \tag{29}$$

Now, using (16a) and (16b), we obtain,

$$\begin{aligned} \left[\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right]\left[\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right] &\leq [\alpha_n^{(1)} + (1-\alpha_n^{(1)}a)][\alpha_n^{(2)} + (1-\alpha_n^{(2)})a] \\ &= [\alpha_n^{(1)} + (1-\alpha_n^{(1)})a][1-(1-\alpha_n^{(2)})(1-a)] \\ &\leq \alpha_n^{(1)} + (1-\alpha_n^{(1)}) a \alpha_n^{(2)} + (1-\alpha_n^{(1)})(1-\alpha_n^{(2)}) a^2. \end{aligned} \tag{30}$$

Similarly, using (16a)-(16c), we get,

$$\begin{aligned} &\left[\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right]\left[\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right]\left[\frac{\alpha_n^{(3)}}{1-(1-\alpha_n^{(3)})a}\right] \\ &\quad \leq [\alpha_n^{(1)} + (1-\alpha_n^{(1)})a][\alpha_n^{(2)} + (1-\alpha_n^{(2)})a][\alpha_n^{(3)} + (1-\alpha_n^{(3)})a] \\ &\quad \leq \alpha_n^{(1)} + (1-\alpha_n^{(1)}) a \alpha_n^{(2)} + (1-\alpha_n^{(1)})(1-\alpha_n^{(2)}) a^2 [\alpha_n^{(3)} + (1-\alpha_n^{(3)})a]. \end{aligned}$$

(31)

Keeping in mind Berinde's definitions (5) and (6), inequalities (16a), (24a) and (25) yield that implicit mann iteration(IM) converges faster than explicit mann iteration(M), inequalities (24b), (28) and (30) yield that implicit Ishikawa iteration (II) converges faster than explicit Ishikawa iteration (I), and inequalities (24c), (29) and (31) yield that implicit Noor iteration (IN) converges faster than explicit Noor iteration (N).

Moreover, again using Berinde's definitions (5) and (6) with

$$\left[\frac{\alpha_n^{(1)}}{1+(1-\alpha_n^{(1)})a}\right]\left[\frac{\alpha_n^{(2)}}{1+(1-\alpha_n^{(2)})a}\right] \leq \left[\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right],$$

$$\left[\frac{\alpha_n^{(1)}}{1+\left(1-\alpha_n^{(1)}\right)a}\right]\left[\frac{\alpha_n^{(2)}}{1-\left(1-\alpha_n^{(2)}\right)a}\right]\left[\frac{\alpha_n^{(3)}}{1-\left(1-\alpha_n^{(3)}\right)a}\right] \leq \left[\frac{\alpha_n^{(1)}}{1-\left(1-\alpha_n^{(1)}\right)a}\right]\left[\frac{\alpha_n^{(2)}}{1-\left(1-\alpha_n^{(2)}\right)a}\right]. \tag{32}$$

And inequalities (24a)-(24d), we conclude that decreasing order of convergence speed of implicit iterations is as follows: implicit k- step, implicit Noor, implicit Ishikawa, and implicit Mann iterations.

Example 14: Let $K = [0,1]$, $T(x) = 1 - x$, $x_0 \neq 0$, $\alpha_n^{(1)} = \alpha_n^{(2)} = \alpha_n^{(3)} = \cdots = \alpha_n^{(k)} = \frac{1}{\sqrt{n}}$, then for implicit Mann iteration (k=1), we have,

$$x_n^{(1)} = \alpha_n^{(1)} x_{n-1}^{(1)} + \left(1-\alpha_n^{(1)}\right)Tx_n^{(1)} = \frac{1}{\sqrt{n}} x_{n-1}^{(1)} + \left(1 - \frac{1}{\sqrt{n}}\right)\left(1 - x_n^{(1)}\right),$$

Which further implies,

$$x_n^{(1)}\left(2 - \frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}} x_{n-1}^{(1)} + 1 - \frac{1}{\sqrt{n}}$$

$$x_n^{(1)} = \frac{x_{n-1}^{(1)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

Also, for implicit Ishikawa iteration (k=2), we have,

$$x_n^{(1)} = \frac{x_{n-1}^{(2)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

$$x_{n-1}^{(2)} = \frac{x_{n-1}^{(1)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

Similarly, for implicit Noor iterations(k=3), we have,

$$x_n^{(1)} = \frac{x_{n-1}^{(2)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

$$x_{n-1}^{(2)} = \frac{x_{n-1}^{(3)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

$$x_{n-1}^{(3)} = \frac{x_{n-1}^{(1)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

For k=4, or 4- step iterations, we have,

$$x_n^{(1)} = \frac{x_{n-1}^{(2)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

$$x_{n-1}^{(2)} = \frac{x_{n-1}^{(3)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

$$x_{n-1}^{(3)} = \frac{x_{n-1}^{(4)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

$$x_{n-1}^{(4)} = \frac{x_{n-1}^{(1)} + \sqrt{n} - 1}{2\sqrt{n} - 1}$$

No.of iterations	K=1	K=2	K=3	K=4
1	1	1	1	1
2	0.77345908034	0.64955973727	0.5917967364	0.54473623001
3	0.61097719293	0.5246318747	0.50546715262	0.5012134535
4	0.5369923974	0.50273687497	0.50020248713	0.50001498097
5	0.51065407522	0.50022701864	0.50000483735	0.50000010307
6	0.50273252918	0.50001493343	0.50000008161	0.50000000045
7	0.50063673017	0.50000081085	0.50000000103	0.5
8	0.50013672967	0.50000003739	0.50000000001	---
9	0.50002734593	0.5000000015	0.5	---
10	0.50000513582	0.50000000005	---	---
11	0.5000009117	0.5	---	---
12	0.50000015379	---	---	---
13	050000002476	---	---	---
14	0.50000000382	---	---	---
15	0.50000000057	---	---	---
16	0.50000000008	---	---	---
17	0.50000000001	---	---	---

18	0.5	---	---	---
19	---	---	---	---

2.3) Data dependence of implicit iterations in hyperbolic spaces

Theorem 15: Let $T: K \to K$ be a mapping satisfying (). Let T_1 be an approximate operator of T as in definition (), and let $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two implicit iterations associated to T, T_1 and defined by (IN).

$$\begin{aligned} u_n^{(1)} &= W\left(u_{n-1}^{(2)}, T_1 u_n^{(1)}, \alpha_n^{(1)}\right), \\ u_{n-1}^{(2)} &= W\left(u_{n-1}^{(3)}, T_1 u_{n-1}^{(2)}, \alpha_n^{(2)}\right), \\ u_{n-1}^{(3)} &= W\left(u_{n-1}^{(4)}, T_1 u_{n-1}^{(3)}, \alpha_n^{(3)}\right), \\ &\dots \\ &\dots \\ &\dots \\ u_{n-1}^{(k)} &= W\left(u_{n-1}^{(1)}, T_1 u_{n-1}^{(k)}, \alpha_n^{(k)}\right). \end{aligned} \tag{33}$$

Respectively, where $\left\{\alpha_n^{(1)}\right\}_{n=0}^{\infty}$, $\left\{\alpha_n^{(2)}\right\}_{n=0}^{\infty}$, $\left\{\alpha_n^{(3)}\right\}_{n=0}^{\infty}$, ..., $\left\{\alpha_n^{(k)}\right\}_{n=0}^{\infty}$ are real sequences in $[0,1]$, satisfying $\sum_{n=0}^{\infty}\left(1-\alpha_n^{(1)}\right) = \infty, \alpha_n^{(1)} \le \alpha_n^{(2)}, \alpha_n^{(1)} \le \alpha_n^{(3)}$. Let $p = Tp$and $q = T_1 q$, then, for $\varepsilon > 0$, we have the following estimate:

$$d(p,q) \le \frac{\varepsilon}{(1-a)^2}. \tag{34}$$

Proof: Using definition (2), iterations (Ik), and iteration(33), we have the following estimates:

$$\begin{aligned} d\left(x_n^{(1)}, u_n^{(1)}\right) &= d\left(W\left(x_{n-1}^{(2)}, Tx_n^{(1)}, \alpha_n^{(1)}\right), W\left(u_{n-1}^{(2)}, T_1 u_n^{(1)}, \alpha_n^{(1)}\right)\right) \\ &\le \alpha_n^{(1)} d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) + \left(1-\alpha_n^{(1)}\right) d\left(Tx_n^{(1)}, T_1 u_n^{(1)}\right) \\ &\le \alpha_n^{(1)} d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) + \left(1-\alpha_n^{(1)}\right)\left\{d\left(Tx_n^{(1)}, T_1 x_n^{(1)}\right) + d\left(T_1 x_n^{(1)}, T_1 u_n^{(1)}\right)\right\} \\ &\le \alpha_n^{(1)} d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) + \left(1-\alpha_n^{(1)}\right)\left\{\varepsilon + \varphi d\left(x_n^{(1)}, T_1 x_n^{(1)}\right) + a d\left(x_n^{(1)}, u_n^{(1)}\right)\right\}. \end{aligned} \tag{35}$$

Which further gives,

$$\left\{1-\left(1-\alpha_n^{(1)}\right)a\right\} d\left(x_n^{(1)}, u_n^{(1)}\right) \le \alpha_n^{(1)} d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) + \left(1-\alpha_n^{(1)}\right)\left\{\varepsilon + \varphi d\left(x_n^{(1)}, T_1 x_n^{(1)}\right)\right\}.$$

$$d\left(x_n^{(1)}, u_n^{(1)}\right) \le \frac{\alpha_n^{(1)}}{1-\left(1-\alpha_n^{(1)}\right)a} d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) + \frac{1-\alpha_n^{(1)}}{1-\left(1-\alpha_n^{(1)}\right)a}\left\{\varepsilon + \varphi d\left(x_n^{(1)}, T_1 x_n^{(1)}\right)\right\}, \tag{36}$$

With,

$$\begin{aligned} d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) &\le d\left(W\left(x_{n-1}^{(3)}, Tx_{n-1}^{(2)}, \alpha_n^{(2)}\right), W\left(u_{n-1}^{(3)}, T_1 u_{n-1}^{(2)}, \alpha_n^{(2)}\right)\right) \\ &\le \alpha_n^{(2)} d\left(x_{n-1}^{(3)}, u_{n-1}^{(3)}\right) + \left(1-\alpha_n^{(2)}\right) d\left(Tx_{n-1}^{(2)}, T_1 u_{n-1}^{(2)}\right), \end{aligned} \tag{37}$$

$$\begin{aligned} d\left(Tx_{n-1}^{(2)}, T_1 u_{n-1}^{(2)}\right) &\le d\left(Tx_{n-1}^{(2)}, T_1 x_{n-1}^{(2)}\right) + d\left(T_1 x_{n-1}^{(2)}, T_1 u_{n-1}^{(2)}\right) \\ &\le \varepsilon + \varphi d\left(x_{n-1}^{(2)}, T_1 x_{n-1}^{(2)}\right) + a d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right), \end{aligned} \tag{38}$$

Using (37) and (38), we have,

$$d\left(x_{n-1}^{(2)}, u_{n-1}^{(2)}\right) \le \frac{\alpha_n^{(2)}}{1-\left(1-\alpha_n^{(2)}\right)a} d\left(x_{n-1}^{(3)}, u_{n-1}^{(3)}\right) + \frac{1-\alpha_n^{(2)}}{1-\left(1-\alpha_n^{(2)}\right)a}\left\{\varepsilon + \varphi d\left(x_{n-1}^{(2)}, T_1 x_{n-1}^{(2)}\right)\right\}, \tag{39}$$

With,

$$d\left(x_{n-1}^{(3)}, u_{n-1}^{(3)}\right) \le \alpha_n^{(3)} d\left(x_{n-1}^{(4)}, u_{n-1}^{(4)}\right) + \left(1-\alpha_n^{(3)}\right) d\left(Tx_{n-1}^{(3)}, T_1 u_{n-1}^{(3)}\right) \tag{40}$$

$$\begin{aligned} d\left(Tx_{n-1}^{(3)}, T_1 u_{n-1}^{(3)}\right) &\le d\left(Tx_{n-1}^{(3)}, T_1 x_{n-1}^{(3)}\right) + d\left(T_1 x_{n-1}^{(3)}, T_1 u_{n-1}^{(3)}\right) \\ &\le \varepsilon + \varphi d\left(x_{n-1}^{(3)}, T_1 x_{n-1}^{(3)}\right) + a d\left(x_{n-1}^{(3)}, u_{n-1}^{(3)}\right) \end{aligned} \tag{41}$$

Using (40) and (41) and simplifying, we get,

$$d\left(x_{n-1}^{(3)}, u_{n-1}^{(3)}\right) \le \frac{\alpha_n^{(3)}}{1-\left(1-\alpha_n^{(3)}\right)a} d\left(x_{n-1}^{(4)}, u_{n-1}^{(4)}\right) + \frac{1-\alpha_n^{(3)}}{1-\left(1-\alpha_n^{(3)}\right)a}\left\{\varepsilon + \varphi d\left(x_{n-1}^{(3)}, T_1 x_{n-1}^{(3)}\right)\right\} \tag{42}$$

In similar lines, we get,

$$d\left(x_{n-1}^{(k)}, u_{n-1}^{(k)}\right) \le \frac{\alpha_n^{(k)}}{1-\left(1-\alpha_n^{(k)}\right)a} d\left(x_{n-1}^{(1)}, u_{n-1}^{(1)}\right) + \frac{1-\alpha_n^{(k)}}{1-\left(1-\alpha_n^{(k)}\right)a}\left\{\varepsilon + \varphi d\left(x_{n-1}^{(k)}, T_1 x_{n-1}^{(k)}\right)\right\} \tag{43}$$

Using these estimates, we arrive at,

$$
\begin{aligned}
&d(x_n^{(1)}, u_n^{(1)}) \\
&\leq \left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left(\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right)\left(\frac{\alpha_n^{(3)}}{1-(1-\alpha_n^{(3)})a}\right)\cdots\left(\frac{\alpha_n^{(k)}}{1-(1-\alpha_n^{(k)})a}\right)d(x_{n-1}^{(1)}, u_{n-1}^{(1)}) \\
&+\left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left(\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right)\cdots\left(\frac{\alpha_n^{(k)}}{1-(1-\alpha_n^{(k)})a}\right)\left\{\varepsilon+\varphi d(x_{n-1}^{(k)}, T_1 x_{n-1}^{(k)})\right\} \\
&+\left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left(\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right)\cdots\left(\frac{\alpha_n^{(k-1)}}{1-(1-\alpha_n^{(k-1)})a}\right)\left\{\varepsilon+\varphi d(x_{n-1}^{(k-1)}, T x_{n-1}^{(k-1)})\right\}+\cdots \\
&+\left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left(\frac{\alpha_n^{(2)}}{1-(1-\alpha_n^{(2)})a}\right)\cdots\left(\frac{\alpha_n^{(k-1)}}{1-(1-\alpha_n^{(k-1)}a)}\right)\left\{\varepsilon+\varphi d(x_{n-1}^{(1)}, T x_{n-1}^{(1)})\right\}
\end{aligned}
$$

(44)

Keeping in mind the inequalities(16a)-(16d), (44) reduces to,

$$
\begin{aligned}
d(x_n^{(1)}, u_n^{(1)}) &\leq [1-(1-\alpha_n^{(1)})(1-a)]d(x_{n-1}^{(1)}, u_{n-1}^{(1)}) + \left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left\{\varepsilon+\varphi\left(d(x_{n-1}^{(k)}, T x_{n-1}^{(k)})\right)\right\} \\
&+\left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left\{\varepsilon+\varphi\left(d(x_{n-1}^{(k-1)}, T x_{n-1}^{(k-1)})\right)\right\}+\cdots \\
&+\left(\frac{\alpha_n^{(1)}}{1-(1-\alpha_n^{(1)})a}\right)\left\{\varepsilon+\varphi\left(d(x_{n-1}^{(1)}, T x_{n-1}^{(1)})\right)\right\} \\
&\leq [1-(1-\alpha_n^{(1)})(1-a)]d(x_{n-1}^{(1)}, u_{n-1}^{(1)}) + \left(\frac{1-\alpha_n^{(1)}}{1-a}\right)\left(\frac{1-a}{1-a}\right)\left\{\varepsilon+\varphi\left(d(x_{n-1}^{(k)}, T x_{n-1}^{(k)})\right)\right\} \\
&+\left(\frac{1-\alpha_n^{(1)}}{1-a}\right)\left(\frac{1-a}{1-a}\right)\left\{\varepsilon+\varphi\left(d(x_{n-1}^{(k-1)}, T x_{n-1}^{(k-1)})\right)\right\}+\cdots \\
&+\left(\frac{1-\alpha_n^{(1)}}{1-a}\right)\left(\frac{1-a}{1-a}\right)\left\{\varepsilon+\varphi\left(d(x_{n-1}^{(1)}, T x_{n-1}^{(1)})\right)\right\}
\end{aligned}
\tag{45}
$$

Putting

$a_n = d(x_n^{(1)}, u_n^{(1)}); r_n = (1-\alpha_n^{(1)})(1-a); and\ t_n = \left\{k\varepsilon+\varphi\left(d(x_{n-1}^{(k)}, T x_{n-1}^{(k)})\right)+\varphi\left(d(x_{n-1}^{(k-1)}, T x_{n-1}^{(k-1)})\right)+\cdots+\varphi\left(d(x_{n-1}^{(1)}, T x_{n-1}^{(1)})\right)\right\}\Big/(1-a)^2,$ the above inequality becomes, $a_n \leq (1-r_n)a_{n-1} + r_n t_n$. (46)

Now from theorem (9), we have $\lim_{n\to\infty} d(x_n^{(1)}, p) = 0, \lim_{n\to\infty} d(u_n^{(1)}, p) = 0$ and since, φ is continuous, hence $\lim_{n\to\infty}\varphi\left(d(x_n^{(1)}, T x_n^{(1)})\right) = \lim_{n\to\infty}\varphi\left(d(x_n^{(2)}, T x_n^{(2)})\right) = \cdots = \lim_{n\to\infty}\varphi\left(d(x_n^{(k)}, T x_n^{(k)})\right) = 0$.

Therefore, using lemma (8), (46) yields,

$d(p,q) \leq \frac{k\varepsilon}{(1-a)^2}$. (47)

Remark 16: Putting $\alpha_n^{(k)} = \alpha_n^{(k-1)} = \cdots = \alpha_n^{(2)} = 1$ in k- step iteration and (33), data dependence result of implicit Mann iteration (IM) can be proved easily on the same lines in theorem (15).

Remark 17: Putting $\alpha_n^{(k)} = \alpha_n^{(k-1)} = \cdots = \alpha_n^{(3)} = 1$ in k- step iteration and (33), data dependence result of implicit Ishikawa iteration (II) can be proved easily on the same lines in theorem (15).

Remark 18: putting $\alpha_n^{(k)} = \alpha_n^{(k-1)} = \cdots = \alpha_n^{(4)} = 1$ in k- step iteration and (33), data dependence result of implicit Noor iteration (IN) can be proved easily on the same lines in theorem (15).

III. Applications

Implicit iterations in RNN (Recurrent Neural Network) Analysis.

Neural networks are a class of nonlinear functions. Approximations and stable states are achieved in recurrent auto associative neural networks using iterations. Here we analyze the convergence speed of implicit iterations in recurrent network and many important results can be drawn. The achieved results possess multifaced real line applications and in particular can be helpful to design the inner product of kernel of support vector machine with faster convergence rate.

IV. Conclusion

(1) The speed of implicit iterations depends on the parameters $\alpha_n^{(1)}, \alpha_n^{(2)}$ and $\alpha_n^{(3)}, \ldots, \alpha_n^{(k)}$.
(2) The k- step iterative scheme is the general case of the all three Noor, Ishikawa and Mann iterations and can be useful to chose number of steps of the iterative schemes according to our need.

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