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A Short Note On Pairwise Fuzzy σ-Baire Spaces

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Abstract: *In this paper we study the inter relations between pairwise fuzzy σ-Baire spaces and other fuzzy bitopological spaces.*
Keywords: *Pairwise fuzzy Baire space, pairwise fuzzy F_σ-set, pairwise fuzzy G_δ-set, pairwise fuzzy open set, pairwise fuzzy submaximal space, pairwise fuzzy σ-nowhere dense set, pairwise fuzzy σ-second category space.*

I. Introduction

In order to deal with uncertainties, the idea of fuzzy sets, fuzzy set operations was introduced by L.A.Zadeh [1] in his classical paper in the year 1965, describing fuzziness mathematically for the first time. Since then, the notation of fuzziness has been applied for the study in all branches of Mathematics. Among the first field of Mathematics to be considered in the context of fuzzy sets was general topology. The concept of fuzzy was defined by C.L.Chang [2] in the year 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then, much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concept of σ-nowhere dense sets in classical topology was introduced and studied by Jiling Cao and Sina Greenwood in [3].

In 1989, A.Kandil [4] introduced and studied fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. The concept of Baire bitopological spaces have been studied extensively in classical topology in [5] ,[6] and [7]. The concept of Pairwise fuzzy σ-nowhere dense set is introduced and studied in [8]. By using pairwise fuzzy σ-nowhere dense sets, the concept of pairwise fuzzy σ-Baire space is defined and studied by authors in [9] and [10]. In this paper the inter relations between pairwise fuzzy σ-Baire spaces and other fuzzy bitopological spaces are investigated.

II. Preliminaries

In order to make the exposition self-contained, we give some basic notations and results used in the sequel. In this work by by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are fuzzy topologies on the non-empty set X. Let X be a non-empty set and I the unit interval [0,1]. A fuzzy set λ in X is a mapping from X into I.

Definition 2.1. [2] Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define :
(i) $\lambda \vee \mu : X \to [0,1]$ as follows : $(\lambda \vee \mu)(x) = \max \{ \lambda (x), \mu(x)\}$.
(ii) $\lambda \wedge \mu : X \to [0,1]$ as follows : $(\lambda \wedge \mu)(x) = \min \{ \lambda (x), \mu(x)\}$.
(iii) $\mu = \lambda^c$, $\mu(x) = 1 - \lambda (x)$.
More generally, for a family of $\lambda_i/i \in I$ of fuzzy sets in X, $\vee_i\lambda_i$ and $\wedge_i\lambda_i$ are defined as $\vee_i \lambda_i = \sup_i\{ \lambda_i (x) / x \in X\}$ and $\wedge_i \lambda_i = \inf_i\{ \lambda_i (x) / x \in X\}$.

Definition 2.2. [2] Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define the interior and the closure of λ respectively as follows :
(i) $\mathrm{int}(\lambda) = \vee\{\mu/\mu \leq \lambda , \mu \in T\}$,
(ii) $\mathrm{cl}(\lambda) = \wedge\{\mu/ \lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3. [11] For a fuzzy set λ in a fuzzy topological space X,
(i) $1 - \mathrm{int}(\lambda) = \mathrm{cl}(1 - \lambda)$,
(ii) $1 - \mathrm{cl}(\lambda) = \mathrm{int}(1 - \lambda)$.

Definition 2.4. [8] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set if $\lambda \in T_i$ (i = 1, 2). The complement of pairwise fuzzy open set in $(X, T_1, T_2)$is called a pairwise fuzzy closed set.

Definition 2.5. [8] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.6. [8] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy F_σ-set if $\lambda= \vee_{i=1}^{\infty} (\lambda_i)$, where (λ_i)'s are pairwise fuzzy closed sets in (X, T_1, T_2).

Lemma 2.7. [11] For a family of $\{\lambda_\alpha\}$ of fuzzy sets of a fuzzy topological space (X, T), $\vee cl(\lambda_\alpha) \le cl(\vee \lambda_\alpha)$. In case *A* is a finite set, $\vee cl(\lambda_\alpha) = cl(\vee \lambda_\alpha)$. Also $\vee\, int(\lambda_\alpha) \le int(\vee \lambda_\alpha)$ in (X, T).
Definition 2.8. [12] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy dense set if $cl_{T1}\, cl_{T2}(\lambda) = cl_{T2}\, cl_{T1}(\lambda) = 1$, in (X, T_1, T_2).
Definition 2.9. [13] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy nowhere dense set if $int_{T1}\, cl_{T2}(\lambda) = int_{T2}\, cl_{T1}(\lambda) = 0$, in (X, T_1, T_2).
Definition 2.10. [8] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy σ-nowhere dense set if λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2) such that $int_{T1} int_{T2}(\lambda) = int_{T2}\, int_{T1}(\lambda) = 0$.
Definition 2.11. [9] Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a pairwise fuzzy σ-first category set if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Any other fuzzy set in (X, T_1, T_2) is said to be a pairwise fuzzy σ-second category set in (X, T_1, T_2).
Definition 2.12. [9] If λ is a pairwise fuzzy σ-first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a pairwise fuzzy σ-residual set in (X, T_1, T_2).
Definition 2.13. [9] A fuzzy bitopological space (X, T_1, T_2) is called pairwise fuzzy σ-first category if the fuzzy set 1_X is a pairwise fuzzy σ-first category set in (X, T_1, T_2). That is $1_X = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Otherwise (X, T_1, T_2) will be called a pairwise fuzzy σ-second category space.
Definition 2.14. [13] A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy Baire space if $int_{Ti}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$, (i=1,2) where (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).
Definition 2.15. [14] A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy almost resolvable space if $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where the fuzzy sets (λ_k)'s in (X, T_1, T_2) are such that $int_{T1}int_{T2}(\lambda_k) = 0 = int_{T2}int_{T1}(\lambda_k)$. Otherwise (X, T_1, T_2) is called a pairwise fuzzy almost irresolvable space.

III. Pairwise fuzzy σ -Baire Spaces

Definition 3.1. [8] A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy σ-Baire space if $int_{Ti}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$, (i = 1, 2) where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2).
Theorem 3.2. [9] Let (X, T_1, T_2) be a fuzzy bitopological space. Then the following are equivalent
(1.) (X, T_1, T_2) is a pairwise fuzzy σ-Baire space.
(2.) $int_{Ti}(\lambda) = 0$, (i = 1, 2) for every pairwise fuzzy σ- first category set λ in (X, T_1, T_2).
(3.) $cl_{Ti}(\mu) = 1$, (i = 1, 2) for every pairwise fuzzy σ- residual set μ in (X, T_1, T_2).
Proposition 3.3. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-first category space then (X, T_1, T_2) is a pairwise fuzzy almost resolvable space.
Proof. Let the fuzzy bitopological space (X, T_1, T_2) be a pairwise fuzzy σ-first category space. Then we have $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Since (λ_k) is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2), (λ_k) is a pairwise fuzzy F_σ-set in (X, T_1, T_2) and $int_{T1}int_{T2}(\lambda_k) = 0 = int_{T2}int_{T1}(\lambda_k)$. Hence $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where $int_{T1}int_{T2}(\lambda_k) = 0 = int_{T2}\, int_{T1}(\lambda_k)$, implies that (X, T_1, T_2) is a pairwise fuzzy almost resolvable space.
Theorem 3.4. [9] If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, then (X, T_1, T_2) is a pairwise fuzzy σ-second category space.
Proposition 3.5. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, then (X, T_1, T_2) is a pairwise fuzzy almost irresolvable space.
Proof. Let (X, T_1, T_2) be a pairwise fuzzy σ-Baire space. Then by theorem 3.4, (X, T_1, T_2) is a pairwise fuzzy σ-second category space,and hence (X, T_1, T_2) is not a pairwise fuzzy σ-first category space. Then $\vee_{k=1}^{\infty}(\lambda_k) \ne 1$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Now (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2), implies that $int_{T1}int_{T2}(\lambda_k) = 0 = int_{T2}int_{T1}(\lambda_k)$. Hence, we have $\vee_{k=1}^{\infty}(\lambda_k) \ne 1$, where $int_{T1}\, int_{T2}(\lambda_k) = 0 = int_{T2}\, int_{T1}(\lambda_k)$ and therefore (X, T_1, T_2) is a pairwise fuzzy almost irresolvable space.
Definition 3.6. [14] A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy strongly irresolvable space if $cl_{T1}\, int_{T2}(\lambda_k) = 1 = cl_{T2}\, int_{T1}(\lambda_k)$ for each pairwise fuzzy dense set λ in (X, T_1, T_2).
Proposition 3.7. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable, pairwise fuzzy σ-Baire space and λ is a pairwise fuzzy σ- first category set in (X, T_1, T_2), then λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).
Proof. Let λ be a pairwise fuzzy σ- first category set in (X, T_1, T_2). Then $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where (λ_k)'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy σ-Baire space. By theorem 3.2, $int_{Ti}(\lambda) = 0$ in (X, T_1, T_2). Then we have $1 - int_{Ti}(\lambda) = 1$. This implies that $cl_{Ti}(1 - \lambda) = 1$. Then $cl_{T1}\, cl_{T2}(1 - \lambda) = 1 = cl_{T2}\, cl_{T1}(1 - \lambda)$. Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, for the pairwise fuzzy dense set $1 - \lambda$ in (X, T_1, T_2), we have $cl_{T1}int_{T2}(1 - \lambda) = 1 = cl_{T2}int_{T1}(1 - \lambda)$.

Then $1- \text{int}_{T1}\, \text{cl}_{T2} (\lambda) = 1 = 1 - \text{int}_{T2}\, \text{cl}_{T1} (\lambda)$ and hence $\text{int}_{T1}\, \text{cl}_{T2} (\lambda) = 0 = \text{int}_{T2}\, \text{cl}_{T1} (\lambda)$. Therefore λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).

Proposition 3.8. If (λ_k)'s are the pairwise fuzzy σ-first category sets in a pairwise fuzzy strongly irresolvable, pairwise fuzzy σ-Baire space (X, T_1, T_2), then $\vee_{k=1}^{\infty} (\lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2).

Proof. Let (λ_k)'s be pairwise fuzzy σ-first category sets in a pairwise fuzzy strongly irresolvable, pairwise fuzzy σ-Baire space (X, T_1, T_2). Then by Proposition 3.7, (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2) and hence $\vee_{k=1}^{\infty} (\lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2).

Proposition 3.9. If $\text{int}_{Ti}(\vee_{k=1}^{\infty} (\lambda_k)) = 0$ where (λ_k)'s are the pairwise fuzzy σ-first category sets in a pairwise fuzzy strongly irresolvable, pairwise fuzzy σ-Baire space in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.

Proof. Let (λ_k)'s be pairwise fuzzy σ-first category sets in a pairwise fuzzy strongly irresolvable, pairwise fuzzy σ-Baire space in (X, T_1, T_2). Then by proposition 3.8, $\vee_{k=1}^{\infty} (\lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2). Then $\vee_{k=1}^{\infty} (\lambda_k) = \vee_{j=1}^{\infty} (\mu_j)$, where the fuzzy sets (μ_j)'s are pairwise fuzzy nowhere dense set in (X, T_1, T_2).Since (X, T_1, T_2) is a pairwise fuzzy σ-Baire space then $\text{int}_{Ti} (\vee_{k=1}^{\infty} (\lambda_k)) = 0$ implies that $\text{int}_{Ti} (\vee_{j=1}^{\infty} (\mu_j)) = 0$, where the fuzzy sets (μ_j)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Hence the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space.

Definition 3.10. [14] A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy submaximal space if each pairwise fuzzy dense set in (X, T_1, T_2), is a pairwise fuzzy open set in (X, T_1, T_2). That is., if λ is a pairwise fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2), then $\lambda \in T_i (i = 1, 2)$.

Proposition 3.11. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal space and $\text{cl}_{Ti} (\lambda) = 1$,where the fuzzy set λ is defined on (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space.

Proof. Let λ be a non zero pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, $\lambda \in T_i$ $(i = 1,2)$. Then $\text{int}_{T1} (\lambda) = \lambda = \text{int}_{T2} (\lambda)$ in (X, T_1, T_2), Now $\text{cl}_{T1}\, \text{int}_{T2} (\lambda) = \text{cl}_{T1} (\lambda) = 1$ and also $\text{cl}_{T2}\, \text{int}_{T1} (\lambda) = \text{cl}_{T2} (\lambda) = 1$. Thus $\text{cl}_{T1}\, \text{int}_{T2} (\lambda) = 1 = \text{cl}_{T2}\text{int}_{T1} (\lambda)$ for a pairwise fuzzy dense set λ in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space.

Proposition 3.12. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal, pairwise fuzzy σ-Baire space and λ is a pairwise fuzzy σ-first category set in (X, T_1, T_2), then λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).

Proof. Let λ be a pairwise fuzzy σ-first category set in a pairwise fuzzy submaximal, pairwise fuzzy σ-Baire space. Since (X, T_1, T_2) is a pairwise fuzzy submaximal space, by proposition 3.11, (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space. Then (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable, pairwise fuzzy σ-Baire space. Since λ is a pairwise fuzzy σ-first category set in (X, T_1, T_2), By proposition 3.7, λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).

Proposition 3.13. If (λ_k)'s are pairwise fuzzy σ-first category sets in a pairwise fuzzy submaximal, pairwise fuzzy σ-Baire space (X, T_1, T_2), then $\vee_{k=1}^{\infty} (\lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2).

Proof. Let (λ_k)'s be pairwise fuzzy σ-first category sets in a pairwise fuzzy submaximal, pairwise fuzzy σ-Baire space in (X, T_1, T_2). Then by proposition 3.12, (λ_k)'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2) and hence $\vee_{k=1}^{\infty} (\lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2).

Proposition 3.14. If $\text{int}_{Ti} (\vee_{k=1}^{\infty} (\lambda_k)) = 0$, where (λ_k)'s are the pairwise fuzzy σ-first category set in a pairwise fuzzy submaximal, pairwise fuzzy σ-Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.

Proof. Let (λ_k)'s be pairwise fuzzy σ-first category sets in a pairwise fuzzy submaximal, pairwise fuzzy σ-Baire space in (X, T_1, T_2). Then by proposition 3.13, $\vee_{k=1}^{\infty} (\lambda_k)$ is a pairwise fuzzy first category set in (X, T_1, T_2). Then $\vee_{k=1}^{\infty} (\lambda_k) = \vee_{j=1}^{\infty} (\mu_j)$, where the fuzzy sets (μ_j)'s are pairwise fuzzy nowhere dense set in (X, T_1, T_2). Now $\text{int}_{Ti} (\vee_{k=1}^{\infty} (\lambda_k)) = 0$ implies that $\text{int}_{Ti} (\vee_{j=1}^{\infty} (\mu_j)) = 0$, where the fuzzy sets (μ_j)'s are pairwise fuzzy nowhere dense set in (X, T_1, T_2). Hence (X, T_1, T_2) is a pairwise fuzzy Baire space.

IV. Conclusion

In this paper, conditions under which pairwise fuzzy σ-first category sets become pairwise fuzzy nowhere dense sets are established. The conditions under which pairwise fuzzy σ-Baire space become pairwise fuzzy Baire spaces are also established. The inter relations between pairwise fuzzy σ-Baire spaces and other fuzzy bitopological spaces are also studied.

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