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General Formula of Linear Mixed Integral Equation with Weak Singular Kernel

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Abstract: *In this paper, we consider a mixed integral equation (**MIE**) of the second kind. Under certain conditions, the existence of a unique solution of is discussed and proved. The kernel of position takes asingular form, while the kernel of time is continuous. Using a quadratic numerical method, the **MIE** leads us to a linear system of Fredholm integral equations (**SFIEs**). Then,**SFIEs**after using Toeplitz matrix method (**TMM**),tends to a linear algebraic system (**LAS**). The existence of a unique solution of **LAS** is proved. Finally, numerical examples are considered, and the error, in each case, is calculated.*

Keywords and phrases: *Mixed integral equation, Linear system of Fredholm integral equations, Linear algebraic system, Carleman function, logarithmickernel, Toeplitz matrix method.*
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I. Introduction

The types of integralequations (**IEs**)arise in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, antenna synthesis problem, mathematical physics and contact problem in the theory of elasticity, see [1-4]

In recent years, the theory of **IEs** has close contact with many different areas of mathematics. The following books contain many different methods to obtain the solution of the integral equation numerically, see [5-8]. The singular **IEs** appear in a variety of applications concerning the problems in the potential theory, see [9],wave scattering in quantum mechanics[10], diffraction problems of aero/hydroacoustis [11]. The common approach to the solution of this type involves its reduction to an equation with Cauchy and Carleman kernel.

Consider a generalized formula of linear integral equation

$$\mu\,\phi(x,t)=\lambda\int_{\Omega} k(|x-y|)\,\phi(y,t)dy+\lambda\int_{0}^{t}\int_{\Omega} F(t,\tau)k(|x-y|)\,\phi(y,\tau)\,dy\,d\tau$$
$$+\lambda\int_{0}^{t} G(t,\tau)\,\phi(x,\tau)d\tau+f(x,t)\ \ ,(x=\bar{x}(x_1,x_2,\dots,x_n)\ \ ,y=\bar{y}(y_1,y_2,\dots,y_n))\,.\quad(1.1)$$

The previous linear **IEs** is considered in the space $L_2(\Omega)\times C[0,T]$, $T<1$, where Ω is the domain of integration with respect to position and $t\in[0,T]$ is the time. Here, in (1.1) the Fredholm integral term has a singular kernel $(|x-y|)$.While the Volterra integral term has the positive and continuous twokernels (t,τ) , and $G(t,\tau)$ for all time $t,\tau\in[0,T],T<1$. The coefficient λ is a constant, has manyphysical meaning,while, μ is a constant define the kind of the integral equation. The given function $f(x,t)$is called the free term, and $\phi(x,t)$ is the unknown function.

Many special cases can be derived from the integral equation(1.1),

(1) If $F(t,\tau)=0$, we have

$$\mu\,\phi(x,t)=\lambda\int_{\Omega} k(|x-y|)\,\phi(y,t)dy+\lambda\int_{0}^{t} G(t,\tau)\,\phi(x,\tau)d\tau+f(x,t)\ \ ,\qquad(1.2)$$

The above formula (1.2) is discussed in [12].

(2) If,$G(t,\tau)=0$, we obtain

$$\mu\,\phi(x,t)=\lambda\int_{\Omega} k(|x-y|)\,\phi(y,t)dy+\lambda\int_{0}^{t}\int_{\Omega} F(t,\tau)k(|x-y|)\,\phi(y,\tau)\,dy\,d\tau+f(x,t)\,.$$
$$(1.3)$$

The solution of the above formula (1.2) is obtained in [13].

(3) If,in (1.3), $\mu=0$, we have a **MIE** of the first kind, where [14] many spectral relationshipsare obtained in [14]

In addition the work of Abdou et al.,in[15-17] is considered special cases of this work
In order to guarantee the existence of a unique solution of the linear equation (1.1), we assume the following conditions:
(i) The kernel ofposition satisfies the discontinuity condition

$$\left[\int_{\Omega}\int_{\Omega} k^2(|x-y|)\,dy\,dx\right]^{1/2} = M, \qquad (M \text{ is a constant}) .$$

(ii) The kernels of time$F(t,\tau)$and $G(t,\tau)$ belong to the class $C[0,T]$, $0 \le \tau \le t \le T < 1$, and satisfy for the constants,$N > N_1$, $N > N_2$, the conditions

$$|F(t,\tau)| \le N_1 \quad , \quad |G(t,\tau)| \le N_2 \quad , \quad \forall\, t\,, \tau \in [0,T] \quad .$$

(iii) The given function $f(t,\tau)$, with its partial derivatives with respect to x and t , are continuous in $L_2(\Omega) \times C[0,T]$, $T < 1$,and its norm is defined as

$$\|f(x,t)\| = \max_{0 \le t \le T} \int_0^t \left[\int_{\Omega} f^2(x,\tau)dx\right]^{\frac{1}{2}} d\tau = H \quad , \quad (H \text{ is a constant}) \ .$$

(iv) The unknown function $\phi(x,t)$ satisfies Lipschitz condition for the first argument and Holder condition for the second argument.

In this paper, the existence of a unique solution of the **IE** (1.1) is discussed and proved. A numerical method is used to translate the **MIE** (1.1) to a system of **FIEs** of the second kind.Then the existence of a unique solution of this system is proved. The **TMM**is used to obtain a **LAS**, where the existence of a unique solution of this system will be proved. Finally, we obtain, numerically the solution of the **LAS** when the kernel takes the weakly forms ($k(|x-y|) = \ln|x-y|$, $k(|x-y|) = |x-y|^{-\upsilon}$, $0 < \upsilon < 1$). Moreover, numerical results are obtained and the error estimate, in each case, is computed.

II. Quadratic Numerical Method

In this section,a quadratic numerical method is used to represent (1.1) as a **SFIEs**. For this, we divide the interval $[0,T]$, $0 \le t \le T < 1$
as $0 = t_0 < t_1 < t_2 < \cdots < t_l < \cdots < t_p = T$, where $t = t_l$, $l = 1,2,\dots,p$; to get

$$\mu\,\phi(x,t_l) = \lambda\int_{\Omega} k(|x-y|)\,\phi(y,t_l)dy + \lambda\int_0^{t_l}\int_{\Omega} F(t_l,\tau)k(|x-y|)\,\phi(y,\tau)\,dy\,d\tau + \lambda\int_0^{t_l} G(t_l,\tau)\,\phi(x,\tau)d\tau + f(x,t_l) \quad . \tag{2.1}$$

Using the quadrature formula,see Atkinson [18] we have

$$\mu_l\phi_l(x) = \lambda_l\int_{\Omega} k(|x-y|)\phi_l(y)\,dy + H_l(x) + E_{p,l}(x) \ , E_{p,l}(x) = \max_{l_1,l_2}\{E_{p,l_1}\,,\,E_{p,l_2}\}\,, \tag{2.2}$$

where$\mu_l = \mu - \lambda\, w_{l_2}\, G_{l,l_2}$, $\lambda_l = \lambda\left(1 + u_l\, F_{l,l_1}\right)$,

and

$$H_l(x) = \lambda\sum_{j=0}^{l_2-1} w_j\; G_{l,j}\;\phi_j(x) + \lambda\sum_{j=0}^{l_1-1} u_j\; F_{l,j}\int_{\Omega} k(|x-y|)\phi_j(y)\,dy + f_l(x) \ , \tag{2.3}$$

Here, we used the following notations:

$$\phi_l(x) = \phi(x,t_l)\,, \quad F_{l,j} = F(t_l,t_j), \quad G_{l,j} = G(t_l,t_j), \quad f_l(x) = f(x,t_l),.$$
$$(l = 0\,,1\,,2\,,\dots,p \ , \ 0 \le j \le l) \qquad (2.4)$$

The characteristic points u_j , w_j and the errors E_{p,l_1},E_{p,l_2}, $p \approx l$, l_1 , $l_2 < l$, are depend on the number of the derivatives of $F(t,\tau)$and $G(t,\tau)$, respectively , to$t \in [0,T]$.
The existence of a unique solution of the (2.2) ,can easily be provedunder the condition (i) and the following conditions

1) $\max_j\left\|f_j(x)\right\|_{L_2(\Omega)} \le Q$, Q is a constant ,

2) For the constants $L > \{ Q_1 , Q_2 \}$, we have $\sum_{j=0}^{l-1} \max_j |u_j F_{l,j}| \le Q_1$, $\sum_{j=0}^{l-1} \max_j |u_j G_{l,j}| \le Q_2$.,

III. The Toeplitz Matrixmethod, See Abdou Et Al. [16, 19]

Write the**SFIEs** (2.2),when $\Omega = [-a, a]$, as

$$\mu_l \phi_l(x) = \psi_l(x) + \lambda_l \int_{\Omega} k(|x-y|)\phi_l(y)\, dy. \qquad (3.1)$$

Write the integral term in the form

$$\int_{-a}^{a} k(|x-y|)\phi_l(y)\, dy = \sum_{n=-N}^{N+1} \int_{nh}^{nh+h} k(|x-y|)\phi_l(y)\, dy \quad , \left(h = \frac{a}{N}\right). \qquad (3.2)$$

Then, approximate the integral in theright hand side of (3.2) by

$$\int_{nh}^{nh+h} k(|x-y|)\phi_l(y)\, dy = A_n^{(l)}(x)\phi_l(nh) + B_n^{(l)}(x)\phi_l(nh+h) + E_{N,n}^{(l)}, \qquad (3.3)$$

where$E_{N,n}^{(l)}$ is the estimate error . Using the principal idea of the **TMM**by assuming in (3.3), $\phi_l(x) = 1 , x$, respectively , in this case $E_{N,n} = 0$.Hence, we have two formulas of two unknown functions $A_n^{(l)}(x)$and $B_n^{(l)}(x)$. By solving the results, the functions $A_n^l(x)$ and $B_n^l(x)$ take the forms

$$A_n^{(l)}(x) = \frac{[(nh+h)I(x) - J(x)]}{h}, \quad B_n^{(l)}(x) = \frac{[J(x) - nh\, I(x)]}{h} \quad , \qquad (3.4)$$

where the values of $I(x)$ and $J(x)$ are

$$I(x) = \int_{nh}^{nh+h} k(|x-y|)\ dy \quad , \quad J(x) = \int_{nh}^{nh+h} y\, k(|x-y|)\ dy \quad , \qquad (3.5)$$

Therefore, the relation (3.2), becomes

$$\int_{-a}^{a} k(|x-y|)\phi_l(y)\, dy = \sum_{n=-N}^{N} G_n^{(l)}(x)\ \phi_l(nh) \quad , \qquad (3.6)$$

where

$$G_n^{(l)}(x) = \begin{cases} A_{-N}^{(l)} & , \ n = -N \\ A_n^{(l)}(x) + B_{n-1}^{(l)}(x) & , \ -N < n < N \\ B_{N-1}^{(l)}(x) & , \ n = N \,;\, 0 \le l \le p \end{cases} \qquad (3.7)$$

The integral equation (3.1), after putting $= m\, h$, becomes

$$\mu_l \phi_{l,m} - \lambda_l \sum_{n=-N}^{N} Y_{n,m}^{(l)} \phi_{l,n} = \psi_{l,m} \quad , \quad l = 0,1,2, \dots, p \quad , \qquad (3.8)$$

The solution of the formula (3.8) takes the form

$$\phi_{l,m} = \left[\mu_l\, I - \lambda_l Y_{n,m}^{(l)}\right]^{-1} \psi_{l,m} \quad , \quad \left|\mu_l\, I - \lambda_l Y_{n,m}^{(l)}\right| \ne 0 \ . \qquad (3.9)$$

The formula (3.8) or (3.9) represents a **SAE**s, whereϕ_m is a vector of $2N+1$elements,I is the unit matrix of order $n \times m$and $\mathrm{Y}_{n,m}$ is a matrix whose elements are given by

$$Y_{n,m}^{(l)} = G_{n-m} + P_{n,m} \quad ,$$

$$G_{n-m} = A_n^{(l)}(mh) + B_{n-1}^{(l)}(mh) \quad , \quad -N \le n \le N \qquad (3.10)$$

The matrix G_{n-m} is a Toeplitz matrix of order $2N+1$, where $-N \le m, n \le N$ and the elements of the second matrix are zeros except the elements of the first and last rows (columns). We can evaluate the values of the first row by substituting in $B_{n-1}^{(l)}(mh)$ $n = -N$and $= -N + i$, $0 \le i \le 2\, N$, and the values of the last row (column) by substituting , in $A_n^{(l)}(mh)$ $n = -N$ and $m = -N + i$.

Definition1: The **TMM** is said to be convergent of order r in $[-a, a]$, if for N sufficiently large , there exist a constant $D > 0$ independent of N such that

$$\|\phi(x) - \phi_N(x)\| \le D\, N^{-r} \quad , \qquad (3.11)$$

The error term $E_{N,n}^{(l)}$ is determined from the following formula

$$E_{N,n}^{(l)} = \left|\int_{\mathrm{nh}}^{nh+h} y^2 k(|x-y|)\ dy - A_n^{(l)}(x)(nh)^2 - B_n^{(l)}(x)(nh+h)^2\right| = O(h_l^3)\ ,(h_l^4 \to 0). \qquad (3.12)$$

The existence of a unique solution of the algebraic systemin ℓ^{∞} :

For thisaim, we write the system (3.8) in the operator form

$$\bar{T}\phi_{l,m} = T\phi_{l,n} + \psi_{l,m} \quad , \quad T\phi_{l,n} = \frac{\lambda_l}{\mu_l}\sum_{n=-N}^{N} Y_{n,m}^{(l)}\phi_{l,n}, \quad \mu_l \neq 0$$

$$\psi_{l,m} = \frac{\lambda_l}{\mu_l}\sum_{j=0}^{l-1} u_j \left[F_{i,j}\sum_{m=-N}^{N} Y_{j,m}^{(l)}\,\phi_{j,n} + G_{i,j}\,\phi_{j,m}\right] + \frac{1}{\mu_l} f_{l,m} \quad , \qquad (3.13)$$

Then, the following lemma can be proved.

Lemma 1: If the kernel of position satisfies the conditions:

$$(i)\colon k(|x-y|) \in L_q \ ,\ (q>1)\ ,(ii)\colon \lim_{x' \to x}\left\|k(|x'-y|) - k(|x-y|)\right\|_{L_q} = 0$$

$$x, x' \in [-a,a] \ , \qquad (3.14)$$

we have

$$a)\ \max_{0\le l\le p}\ \mathrm{Sup}_N \sum_{n=-N}^{N} \left|Y_{n,m}^{(l)}(x)\right| < c\ , c \text{ is a constant} \quad ,$$

$$b)\ \lim_{m' \to m}\ \max_{0\le l\le p}\ \mathrm{Sup}_N \sum_{n=-N}^{N} \left|Y_{n,m'}^{(l)} - Y_{n,m}^{(l)}\right| = 0\ , \quad \forall\ \ 0 \le l \le p\ . \qquad (3.15)$$

Proof: Firstly, we go to prove (a) of Eq. (3.15). S0, from the first formula of (3.4), we obtain

$$\left|A_n^{(l)}(x)\right| = \left[(nh+h)\int_{\mathrm{nh}}^{nh+h} |k(|x-y|)|\ dy - \int_{\mathrm{nh}}^{nh+h} |y||k(|x-y|)|\ dy\right] \Big/ |h| \quad .$$

Then, using the first condition (i) of Eq. (3.14), we deduce that there exist a small constant z_1 , such that $\max_{0\le l\le p} \sum_{n=-N}^{N} \left|A_n^{(l)}(x)\right| \le z_1, \forall N$. Since each term in this inequality is boundedabove, hence for $x = mh$, we write

$$\max_{0\le l\le p}\ \mathrm{Sup}_N \sum_{n=-N}^{N} \left|A_n^{(l)}(mh)\right| \le z_1 . \qquad (3.16)$$

Similarly, from the second formula of (3.4), we can find a small constant z_2 , such that

$$\max_{0\le l\le p}\mathrm{Sup}_N \sum_{n=-N}^{N} \left|B_n^{(l)}(mh)\right| \le z_2 . \qquad (3.17)$$

Therefore,from the relations (3.7), (3.16) and (3.17),there exists a small constant$z \le (z_1 + z_2)$, such that the first inequality of (3.15) is proved.

To prove (b), using the first formula of (3.4), for $x, x' \in [-a,a]$, and applying Hölder inequality , then summing from $n = -N$ to $n = N$, to have

$$\max_{0\le l\le p} \sum_{n=-N}^{n=N} \left|A_n^{(l)}(x') - A_n^{(l)}(x)\right| \le \sum_{n=-N}^{n=N} q_n \left\|k(|x'-y|) - k(|x-y|)\right\|_{L_q} \quad .$$

Putting $= mh$, $x' = m'h$, then using the condition (3.14), when $x' \to x$, we get

$$\lim_{m' \to m}\ \max_{0\le l\le p}\ \mathrm{Sup}_N \sum_{n=-N}^{n=N} \left|A_n^{(l)}(m'h) - A_n^{(l)}(mh)\right| = 0 \quad . \qquad (3.18)$$

Similarly, in view of the second formula of (3.4) , we have

$$\lim_{m' \to m}\ \max_{0\le l\le p}\ \mathrm{Sup}_N \sum_{n=-N}^{n=N} \left|B_n^{(l)}(m'h) - B_n^{(l)}(mh)\right| = 0 \quad . \qquad (3.19)$$

Hence, from (3.18) and (3.19) ,the second inequality of (3.15) is proved.◘

Now, under the two conditions of lemma 3.1 and the following condition

$(c) \max_{0\le l\le p} \operatorname{Sup}_{m} \sum_{n=-N}^{N} |f_{n,m}| \le H$, (H is a constant)

We can state the following

Theorem 1: The **LAS**of (3.8) or (3.13),has a unique solution in the Banachspace ℓ^{∞}, under the following condition

$$|\lambda_l| < |\mu_l|/(c + Lc + L) \ , \ (L \text{ is defined by condition 2}) \ . \qquad (3.20)$$

To prove thistheorem, we consider the following twolemmas.

Lemma 2: The operator $\bar{T}$ of Eq. (3.13) is bounded.

Proof: Let$V$ be the set of all sequences$\Phi = \{\phi_{l,m}\}$ in ℓ^{∞} such that$\|\Phi\|_{\ell^{\infty}} \le \beta$, β is aconstant. Define the norm of the operator $\bar{T}$ Φin Banach space ℓ^{∞} by

$\|\bar{T}\,\Phi\| = \max_{0\le l\le p} \operatorname{Sup}_{m} |\bar{T}\phi_{l,m}|$, for each integer m. (3.21)

Hence, the integral operator (3.13) takes the form

$$\max_{0\le l\le p} \operatorname{Sup}_{m} |\bar{T}\phi_{l,m}| \le \left|\frac{\lambda_l}{\mu_l}\right| \left\{ \max_{0\le l\le p} \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \operatorname{Sup}_{n} |\phi_{l,n}| + \right.$$
$$\left. \max_{0\le l\le p} \sum_{j=0}^{l} \left[|u_j F_{i,j}| \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \operatorname{Sup}_{n} |\phi_{l,n}| + |u_j G_{l,j}| \operatorname{Sup}_{m} |\phi_{l,m}| \right] \right\} + \left|\frac{1}{\mu_l}\right| \max_{0\le l\le p} \operatorname{Sup}_{m} |f_{l,m}|$$

Using condition (2), and Lemma 3.1 and condition (c), we have

$$\max_{0\le l\le p} \operatorname{Sup}_{m} |\bar{T}\phi_m| \le \left|\frac{\lambda_l}{\mu_l}\right| \left\{ \max_{0\le l\le p} \operatorname{Sup}_{n} \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \|\Phi\|_{\ell^{\infty}} + \right.$$
$$\left. \sum_{j=0}^{l} \left[L \max_{0\le l\le p} \operatorname{Sup}_{n} \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \|\Phi\|_{\ell^{\infty}} + L \|\Phi\|_{\ell^{\infty}} \right] \right\} + \left|\frac{1}{\mu_l}\right| H$$

Finally, with the aid of (3.21), we have

$$\|\mathrm{T}\,\Phi\| = \max_{0\le l\le p} \operatorname{Sup}_{m} |\bar{T}\phi_{l,m}| \le \alpha_2 \|\Phi\|_{\ell^{\infty}} + \frac{H}{|\mu_l|} \ , \ \left(\alpha_2 = |\lambda_l/\mu_l|(c + Lc + L)\right). \quad (3.22)$$

The inequality (3.22) shows that the operator $\bar{T}$ maps the set V into itself, where $\beta = H/|\mu_l|(1-\alpha_2)$.

Hence, we deduce $\alpha_2 < 1$. In addition, the inequality (3.22) involves the boundedness of the operators T and $\bar{T}$.

Lemma 3: The operator $\bar{T}$ is continuous and contractive operator.

Proof: For the two sets $\Phi = \{\phi_{l,m}\}$, and , $\bar{\Phi} = \{\overline{\phi_{l,m}}\}$, we have

$$\max_{0\le l\le p} \operatorname{Sup}_{m} |\bar{T}\phi_{l,m} - \bar{T}\overline{\phi_{l,m}}| \le \left|\frac{\lambda}{\mu}\right| \left\{ \max_{0\le l\le p} \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \operatorname{Sup}_{n} |\phi_{l,n} - \overline{\phi_{l,n}}| + \max_{0\le l\le p} \sum_{j=0}^{l} \left[|u_j F_{i,j}| \right.\right.$$
$$\left.\left. \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \operatorname{Sup}_{n} |\phi_{l,n} - \overline{\phi_{l,n}}| + |u_j G_{l,j}| \operatorname{Sup}_{m} |\phi_{l,m} - \overline{\phi_{l,m}}| \right] \right\} \le \left|\frac{\lambda}{\mu}\right| \left\{ \max_{0\le l\le p} \operatorname{Sup}_{n} \right.$$
$$\left. \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \|\Phi - \bar{\Phi}\| + \sum_{j=0}^{l} \left[L \max_{0\le l\le p} \operatorname{Sup}_{n} \sum_{n=-N}^{N} |Y_{n,m}^{(l)}| \|\Phi - \bar{\Phi}\| + L \|\Phi - \bar{\Phi}\| \right] \right\} .$$

Hence, we get

$$\|\bar{T}\,\Phi - \bar{T}\bar{\Phi}\|_{\ell^{\infty}} \le \alpha_2 \|\Phi - \bar{\Phi}\|_{\ell^{\infty}} \quad , \quad \alpha_2 < 1 \quad . \qquad (3.23)$$

The previous inequality tells us that$\bar{T}$ is a continuous operator and under condition (3.20) it is a contractive operator in ℓ^{∞} .Hence, the Theorem is proved.◘

Definition 2: The estimate local error $E_{s,n}$of (3.3) is determined as

$$\phi_l(x) - \left(\phi_l(x)\right)_s = \sum_{n=-s}^{s} Y_{n,m}^{(l)} \left[\phi_l(nh) - \phi_{l,s}(nh)\right] + E_{s,n}^{(l)} \quad , \quad (x = mh) \qquad (3.24)$$

where $\left(\phi_l(x)\right)_s$ is the approximate solution of equation (3.1) .

Corollary1: Assume the hypothesis of theorem 2are verified , then $\lim_{s\to\infty} E_{s,n}^{(l)} = 0$.

Proof: In view of formula (3.23), we have

$$\left|E_{s,n}^{(l)}\right| \leq \max_{0\leq l\leq p} \operatorname{Sup}_{m} \left|\phi_l(mh) - \left(\phi_l(mh)\right)_s\right| + \|\Phi - \Phi_s\|_{\ell^\infty} \operatorname{Sup}_{n} \sum_{n=-s}^{s} \left|Y_{n,m}^{(l)}\right| .$$

The above inequality is true for each integer , and by condition (b) , we get

$$\left\|E_{s,n}^{(l)}\right\|_{\ell^\infty} \leq (1+c)\|\Phi - \Phi_s\|_{\ell^\infty} \quad , \quad \text{for each } s . \qquad (3.25)$$

Since $\|\Phi - \Phi_s\| \to \infty$ as $s \to \infty$ then $E_{s,n}^{(l)} \to 0$ as $s \to \infty$. ◘

Finally, it is convenient to consider the following theorem, which proves the convergence of the sequence of approximate solution $\left\{\left(\phi_l(mh)\right)_{z,s}\right\}$ to the exact solution of equation (1.1) in Banachspace $L_2(\Omega)\times C[0,T]$.

Definition 3: The following relationdetermines the totalerror $E_{Z,S}$:

$$E_{z,s} = \int_{-a}^{a} k(|x-y|)\,\phi(y,t)dy + \int_{0}^{t}\int_{-a}^{a} F(t,\tau)k(|x-y|)\,\phi(y,\tau)\,dy\,d\tau +$$

$$\int_{0}^{t} G(t,\tau)\,\phi(x,\tau)d\tau - \sum_{j=0}^{l} u_l\left[F_{i,j}\sum_{n=-s}^{s} Y_{n,m}^{(l)}\,\phi_{j,n} + G_{j,l}\phi_{j,m}\right] - \sum_{n=-s}^{s} Y_{n,m}^{(l)}\,\phi_{j,n} \;. (4.27)$$

When $z,s\to\infty$, the sum $\sum_{l=0}^{i} u_l\left[F_{i,l}\sum_{n=-s}^{s} Y_{n,m}\,\phi_{l,n} + G_{i,l}\phi_{l,m}\right] + \sum_{n=-s}^{s} Y_{n,m}\,\phi_{l,n} \to$

$$\int_{-a}^{a} k(|x-y|)\,\phi(y,t)dy + \int_{0}^{t}\int_{-a}^{a} F(t,\tau)k(|x-y|)\,\phi(y,\tau)\,dy\,d\tau + \int_{0}^{t} G(t,\tau)\,\phi(x,\tau)d\tau \;,$$

and the solution of the algebraic system becomes the solution of the equation (1.1) .

IV. Application

Consider the mixed integral equation

$$\mu\phi(x,t) = \lambda\int_{\Omega} \phi(y,t)k(|x-y|)dy + \lambda\int_{0}^{t}\int_{\Omega} t^2\tau\,\phi(y,\tau)k(|x-y|)dyd\tau$$

$$+\lambda\int_{\Omega} t\tau^2\phi(x,\tau)d\tau + f(x,t) \;. \quad (\phi(y,t) = x^2 + t^2) \qquad . \qquad (4.1)$$

Example(1): Consider $k(|x-y|) = |x-y|^{-\nu}; 0<\nu<1$, at $\lambda = 0.5152;\ \mu = 1; T = 0.2;\ n = 41$

The approximate solution and the estimate error, in each cases, $for\ \nu = 0.12\ and\ \nu = 0.73$ are calculated in table (1).

Table1

X	Exact	$\upsilon = 0.12$		$\upsilon = 0.73$	
		App. Sol.	Error	App. Sol.	Error
-1	4.00000E-04	3.63253E-04	3.67468E-05	3.08591E-04	9.14093E-05
-0.8	2.56000E-04	2.73278E-04	1.72775E-05	2.24550E-04	3.14502E-05
-0.6	1.44000E-04	1.61339E-04	1.73390E-05	1.10959E-04	3.30406E-05
-0.4	6.40000E-05	8.14052E-05	1.74052E-05	3.00103E-05	3.39897E-05
-0.2	1.60000E-05	3.34485E-05	1.74485E-05	-1.85140E-05	3.45140E-05
0	0.00000E+00	1.74632E-05	1.74632E-05	-3.46828E-05	3.46828E-05
0.2	1.60000E-05	3.34485E-05	1.74485E-05	-1.85140E-05	3.45140E-05
0.4	6.40000E-05	8.14052E-05	1.74052E-05	3.00103E-05	3.39897E-05
0.6	1.44000E-04	1.61339E-04	1.73390E-05	1.10959E-04	3.30406E-05
.8	2.56000E-04	2.73278E-04	1.72775E-05	2.24550E-04	3.14502E-05
1	4.00000E-04	3.63253E-04	3.67468E-05	3.08591E-04	9.14093E-05

(Table (1) describes the exact and numerical solution of Eq. (4.1) when $k(|x-y|) = |x-y|^{-\nu}; 0<\nu<1$, at $\lambda = 0.5152;\ \mu = 1; T = 0.2;\ n = 41;\ for\ \nu = 0.12\ and\ \nu = 0.73$)

We can deduce from the above and other results the following:

1) When the values of λ and T are fixed , the error increases with increasing of υ , where ν is called Poisson ratio , in the theory of elasticity and when $0<\nu<0.5,$ the atomic bond between the particles of the material is normal, while $\upsilon \geq 0.5$ the atomic bond is strong , for this the error may be large .
2) It was found that the highest error value is obtained when $\nu = 0.73$ at $x = \pm 1.$ Also, the errordecreases gradually, and has less valuewhen $= 0.07$.
3) When the values of λ and υ are fixed,the error value increases with the time.
4) The maximum error is 0. 2363917451, at $x = \pm 1$, when $\nu = 0.73, T = 0.8$.
5) The minimum error is 1.81871E-7, at $x = \pm 0.0$, when $\upsilon = 0.07$, $T = 0.004$.
6) In all the studied situations, the error value increases when it get closer to the ends points $x = \pm 1$. It also decreases at the middle when it gets closer to zero.

Example (2): Consider the logarithmic form $k(|x-y|) = \ln|x-y|$,at $\mu = 1,$ $\lambda = .25$ $n = 41,$ $T = 0.2.$

Table 2

x	Exact	App.	Error
-1	4.00E-04	4.14364E-04	1.43636E-05
-0.8	2.56E-04	2.53924E-04	2.07618E-06
-0.6	1.44E-04	1.43201E-04	7.98768E-07
-0.4	6.40E-05	6.37610E-05	2.38987E-07
-0.2	1.60E-05	1.60212E-05	2.12470E-08
0	0.00E+00	9.86122E-08	9.86122E-08
0.2	1.60E-05	1.60212E-05	2.12433E-08
0.4	6.40E-05	6.37610E-05	2.38987E-07
0.6	1.44E-04	1.43201E-04	7.98770E-07
0.8	2.56E-04	2.53924E-04	2.07618E-06
1	4.00E-04	4.14364E-04	1.43636E-05

(Table (2) describes the exact and numerical solution of Eq. (4.1) when $k(|x-y|) = \ln|x-y|$, at $\lambda = 0.25;$ $\mu = 1; T = 0.2;$ $n = 41.$)

We notice from the results of the program at the previous and others cases that

1) When the values of T are fixed,the error values clearly increase with increasing of λ.
2) When the values of λ are fixed,the error values increase with the time increase.
3) The maximum error is 0.020521631, at $x = \pm 0.8$, when $\lambda = 0.4368, T = 0.8.$
4) The minimum error is 8.4973E-10, at $= \pm 0.2$, when $\lambda = 0.25$, $T = 0.004$.
5) The error value increases when it get closer to the ends points $x = \pm 1$. It decreases at the middle when it gets closer to zero.

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