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Additive Results on Generalized Drazin Inverse in Minkowski Space *M*

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Abstract: *In this paper we study the additive properties of generalized Drazin inverse of two Drazin invertible operators in Minkowski spaceM in terms of PQ = QP: Further we have derived the explicit representations of generalized Drazin inverse (P + Q)yG interms of P; Q; PyG and QyG in Minkowski space M .*
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I. Introduction

Throughout we shall deal with $C^{n\times n}$, the space $n \times n$ complex matrices. Let C^n be the space of complex n-tuples, we shall index the components of a complex vector in C^n from 0 to $n-1$, that is $u = (u_0, u_1, u_2, \cdots, u_{n-1})$. Let G be the Minkowski metric tensor defined by $G_u = (u_0, -u_1, -u_2, \cdots, -u_{n-1})$. Clearly the Minkowski metric matrix

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -1_{n-1} \end{bmatrix}, \; G = G^* \text{ and } G^2 = I_n \tag{1.1}$$

In [13], Minkowski inner product on C^n is defined by $(u, v) = [u, G_v]$, where $[\cdot, \cdot]$ denotes the conventional Hilbert (unitary) space inner product. A space with Minkowski inner product is called a Minkowski space and denoted as $\mathscr{M}$.

For $A \in C^{n\times n}$, $x, y \in C^n$, by using (1.1)

$$\begin{aligned}(Ax, y) &= [Ax, Gy] \\ &= [x, A^*Gy] \\ &= [x, G(GA^*G)y] \\ &= [x, G\tilde{A}y], \text{ where } \tilde{A} = GA^*G \\ &= (x, \tilde{A}y)\end{aligned} \tag{1.2}$$

The matrix $\tilde{A}$ is called the Minkowski adjoint of A in $\mathscr{M}$ (A^* is usual Hermitian adjoint of A). Naturally we call a matrix $A \in C^{n\times n}$ m-symmetric in $\mathscr{M}$ if $A = \tilde{A}$. From the definition $\tilde{A} = GA^*G$ we have the following equivalence: A is m-symmetric $\Leftrightarrow$ AG is hermitian $\Leftrightarrow$ GA is hermitian.

For $A \in C^{n\times n}$, $rk(A)$, $N(A)$ and $R(A)$ are respectively the rank of A, null space of A and range space of A. By a generalized inverse of A we mean a solution of the equation $AXA = A$ and is denoted as $A^{(1)} \cdot A\{1\}$ is the set of all generalized inverses of A. Throughout I refers to identity matrix of appropriate order unless otherwise specified.

Definition 1.1 [1]

For $A \in C^{m\times n}$, $A^{\dagger}$ is the Moore-Penrose inverse of A if $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$, $AA^{\dagger}$ and $A^{\dagger}A$ are Hermitian. The Minkowski inverse of A, analogous to Moore-Penrose inverse of A is introduced and its existence is discussed in [12].

Definition 1.2 [12]

For $A \in C^{m\times n}$, A^{m} is the Minkowski inverse of A if $AA^{m}A{=}A$, $A^{m}AA^{m}{=} A^{m}$, AA^{m} and $A^{m}A$ are m-symmetric.

Lemma 1.3

For $A \in C^{n\times n}$ then $A^{\dagger}G$ is the Moore-Penrose inverse of GA in Minkowski's space $\mathscr{M}$, where G is the Minkowski metric tensor of order n.

For $A \in B(X)$, if there exist an operator $A^{\dagger}G \in B(X)$ satisfying the following three operator equations [9].

$$AA^{\dagger}G = A^{\dagger}GA,\ A^{\dagger}GAA^{\dagger}G = A^{\dagger}G,\ A^{k+1}A^{\dagger}G = A^{k}. \tag{1.3}$$

Then $A^{\dagger}$ is called Drazin inverse of A. The smallest k such that $rk(A^{k+1}) = rk(A^{k})$ holds is called index of A, denoted by ind(A). Notice also that ind(A) (if it finite) is the smallest non-negative integers k such that $R(A^{k+1}) {=} R(A^{k})$ and $N(A^{k+1}) {=} N(A^{k})$.

The conditions (1.3) are equivalent to $AA^{\dagger}G = A^{\dagger}GA$, $A^{\dagger}GAA^{\dagger}G = A^{\dagger}G$, $A - A^{2}A^{\dagger}G$ is nilpotent. (1.4)

The concept of generalized Drazin inverse on an infinite-dimensional Banach space was introduced by Koliha [11], which is the element $A^{m} \in B(x)$ such that

$$AA^{m} = A^{m}A,\ A^{m}AA^{m} = A^{m},\ A - A^{2}A^{m} \text{ is quasi nilpotent.} \tag{1.5}$$

If A is generalized Drazin invertible, then the spectral idempotent P of A corresponding to $\{0\}$ is given by $P = I - AA^{m}$. The operator matrix form of A with respect to the space decomposition $X = N(P) \oplus R(P)$ is given by $A = A_1 \oplus A_2$, where A_1 is invertible and A_2 is quasi-nilpotent [1].

In recent years, the characterizations of the Drazin inverses of matrices or operators on an infinite-dimensional space have been considered by many authors (cf.[2-4]), Castro - Gonzalez et al, [2], Djordjevic and Wei [8], Hartwig et al, [10] have studied the generalized Drazin inverse on a Banach space. some additive properties and the explicit expression for the GD-inverse of the sum, are obtained in [3,4,5,7].

In this paper, using the technique of block operator matrices, we will investigate explicit representations of the generalized Drazin inverse $(P+Q)^{\dagger}G$ in term of P, $P^{\dagger}G$, Q and $Q^{\dagger}G$ under the condition of $PQ = QP$. Our results are improvement over the main results of [6]. Indeed, a totally new approach is provided to express the GD-inverse.

This paper is organized as follows. An explicit formula for $(P+Q)^{\dagger}G$ is presented in section 2. This is a key step of the paper. In section 3, special cases are given to indicate the various applications of our main results.

II. Main Result

In the first part of this section, we give new expression for the GD-inverse of $P+Q$ in term of P, $P^{\dagger}G$, Q and $Q^{\dagger}G$. It is interesting to note that our result is quite different from the expression for $(P+Q)^{\dagger}G$ in [8].

Theorem 2.1

Let $P, Q \in B(x)$ be GD-invertible in Minkowski space $\mathscr{M}$ and $PQ = QP$. Then $P+Q$ is GD-invertible if and only if $I + P^{\dagger}GQ$ is GD-invertible in $\mathscr{M}$. In this case we have,

$$(P+Q)^{\dagger}G = P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I - QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger}G)^{n+1}\right]G + \left[\sum_{n=0}^{\infty}(Q^{\dagger}G)^{n+1}(-P)^n\right]G(I - PP^{\dagger}G)$$

and

$$(P+Q)(P+Q)^{\dagger}G = (I+P^{\dagger}Q)^{\dagger}G(PP^{\dagger}+QP^{\dagger})[I+(QQ^{\dagger}G)] + (I+Q^{\dagger}P)^{\dagger}(PQ^{\dagger}+QQ^{\dagger})(I-PP^{\dagger}G).$$

Proof:

Since P is GD-invertible, so we can write $P = P_0 \oplus P_{00}$, where P_0 invertible, P_{00} is Quasi - Nilpotent.

Also $PQ = QP$, we can decompose $Q = Q_0 \oplus Q_{00}$, where Q_0 and Q_{00} are GD-invertible such that $P_0Q_0 = Q_0P_0$ and $P_{00}Q_{00} = Q_{00}P_{00}$.

In a similar way, we can conclude that

$$P_0 = P_1 \oplus P_2, \qquad P_{00} = P_3 \oplus P_4 \text{ and}$$
$$Q_0 = Q_1 \oplus Q_2, \qquad Q_{00} = Q_3 \oplus Q_4,$$

where $P_i (i = 1, 2), Q_j (j = 1, 3)$ are invertible, $P_m (m = 3, 4)$, $Q_n (n = 2, 4)$ are quasi-Nilpotent and $P_i Q_i = Q_i P_i \quad (i = 1, 2, 3, 4)$.

We have $(P + Q) = (P_1 + Q_1) \oplus (P_2 + Q_2) \oplus (P_3 + Q_3) \oplus (P_4 + Q_4)$.

Since P_2 is invertible, Q_2 is Quasi nilpotent and $P_2 Q_2 = Q_2 P_2$,

we have $\rho(P_2 + Q_2) \subset \rho(P_2^\dagger)\rho(Q_2) = \{0\}$.

Thus $P_2^\dagger Q_2$ is Quasi-Nilpotent and $I + P_2^\dagger G Q_2$ is invertible and

$$\begin{aligned}
0 \oplus (P_2 + Q_2)^\dagger G \oplus 0 \oplus 0 &= 0 \oplus P_2^\dagger (I + P_2^\dagger G Q_2)^\dagger G \oplus 0 \oplus 0 \\
&= 0 \oplus P_2^\dagger (I + P_2^\dagger G Q_2)^\dagger G \oplus 0 \oplus 0 \\
&= 0 \oplus P_2^\dagger \left[\sum_{n=0}^{\infty} (P_2^\dagger)^n (-Q_2)^n \right] G \oplus 0 \oplus 0 \\
&= (I - QQ^\dagger G) \left[\sum_{n=0}^{\infty} (P^\dagger)^{n+1} (-Q)^n \right] G.
\end{aligned}$$

Similarly we can prove that $P_3 Q_3^\dagger$ is quasi-Nilpotent and

$I + P_3 Q_3^\dagger$ is invertible with

$$\begin{aligned}
0 \oplus 0 \oplus (P_3 + Q_3)^\dagger 0 &= 0 \oplus 0 \oplus Q_3^\dagger (I + Q_3^\dagger P_3)^\dagger G \oplus 0 \\
&= 0 \oplus 0 \oplus Q_3^\dagger \left[\sum_{n=0}^{\infty} (Q_3^\dagger)^n (-P_3)^n \right] \oplus 0 \\
&= \left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1} (-P)^n \right] G(I - PP^\dagger G).
\end{aligned}$$

Since P_4 and Q_4 are Quasi-Nilpotent and $P_4 Q_4 = Q_4 P_4$,

we get that $P_4 + Q_4$ is GD-invertible and $(P_4 + Q_4)^\dagger G = 0$.

Hence $P + Q$ is GD-invertible, if $P_1 + Q_1$ is GD-invertible.

Note that P_1, $P_1^\dagger G$, Q_1, $Q_1^\dagger G$, $P_1 + Q_1$ and $(P_1 + Q_1)$ commute.

It is easy to know $(P_1 + Q_1)^\dagger G$ is GD-Invertible, if $I + P^\dagger Q$ is GD-Invertible.

Thus
$$\begin{aligned}
(P_1 + Q_1)^\dagger G \oplus 0 \oplus 0 \oplus 0 &= P_1^\dagger (I + P_1 + Q_1)^\dagger G \oplus 0 \oplus 0 \\
&= P_1^\dagger (I + P_1 + Q_1)^\dagger G Q_1 Q_1^\dagger G.
\end{aligned}$$

and

$$\begin{aligned}
(P_1 + Q_1)(P_1 + Q_1)^\dagger G &= (P_1 + Q_1) P_1^\dagger (I + P_1^\dagger Q_1)^\dagger G Q_1 Q_1^\dagger \\
&= (P + Q) P^\dagger (I + P^\dagger Q)^\dagger G Q Q^\dagger G.
\end{aligned}$$

Hence we arrive at

$$(P + Q)^\dagger G = P^\dagger (I + P^\dagger Q)^\dagger G Q Q^\dagger G + (I - QQ^\dagger G) \left[\sum_{n=0}^{\infty} (P^\dagger)^{n+1} (-Q)^n \right] G + \left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1} (-P)^n \right] G(I - PP^\dagger G)$$

Now

$$\begin{aligned}
P^\dagger (I + P^\dagger Q)^\dagger G Q Q^\dagger G &= PP^\dagger (I + P^\dagger Q)^\dagger G Q Q^\dagger G + QP^\dagger (I + P^\dagger Q)^\dagger G Q Q^\dagger G \\
&= (PP^\dagger + QP^\dagger)(I + P^\dagger Q)^\dagger G Q Q^\dagger G \qquad (1.6)
\end{aligned}$$

$$\begin{aligned}
(I - QQ^\dagger G) \left[\sum_{n=0}^{\infty} (-Q)^n (P^\dagger)^{n+1} \right] G &= P(I - QQ^\dagger G) P^\dagger (I + P^\dagger Q)^\dagger G \cdot G + QP^\dagger (I - QQ^\dagger G)(I + P^\dagger Q)^\dagger G \cdot G \\
&= (I - QQ^\dagger G) PP^\dagger (I + P^\dagger Q) + QP^\dagger (I - QQ^\dagger G)(I + P^\dagger Q)^\dagger \\
&= (I - QQ^\dagger G)(PP^\dagger + QP^\dagger)(I + P^\dagger Q)^\dagger . (1.7)
\end{aligned}$$

Similarly,

$$\left[\sum_{n=0}^{\infty}(-P)^n(Q^{\dagger})^{n+1}\right]G(I-PP^{\dagger}G) = PQ^{\dagger}(I+Q^{\dagger}P)^{\dagger}G\cdot G(I-PP^{\dagger}G) + QQ^{\dagger}(I+Q^{\dagger}P)^{\dagger}G\cdot G\cdot(I-PP^{\dagger}G)$$

$$= (I+Q^{\dagger}P)[PQ^{\dagger}+QQ^{\dagger}](I-PP^{\dagger}G) \quad (1.8)$$

$$= (PP^{\dagger}+QP^{\dagger})(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I-QQ^{\dagger}G)(PP^{\dagger}+QP^{\dagger})(I+P^{\dagger}Q) + (I+Q^{\dagger}P)(PQ^{\dagger}+QQ^{\dagger})(I-PP^{\dagger}G)$$

$$(P+Q)(P+Q)^{\dagger}G = (I+P^{\dagger}Q)^{\dagger}G(PP^{\dagger}+QP^{\dagger})[I+(QQ^{\dagger}G)] + (I+Q^{\dagger}P)^{\dagger}(PQ^{\dagger}+QQ^{\dagger})(I-PP^{\dagger}G)$$

Corollary 2.2:

If P and Q are GD-invertible in Minkowski space $\mathscr{M}$ and $PQ = 0$, then $(P+Q)$ is GD-invertible in $\mathscr{M}$ and $(P+Q)^{\dagger}G = P + (I-G)P + Q(I-G)$.

Proof:

$$(P+Q)^{\dagger}G = P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I-QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G + \left[\sum_{n=0}^{\infty}(-P)^n(Q^{\dagger})^{n+1}\right]G(I-QQ^{\dagger}G)$$

Since $PQ = QP = 0$, so we have $P^{\dagger}(P^{\dagger}Q)^{\dagger}GQQG = P + 0 = P$

$$(I-QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G = (1-G)P;$$

$$\left[\sum_{n=0}^{\infty}(-P)^n(Q^{\dagger})^{n+1}\right]G(I-PP^{\dagger}G) = Q(I-G)$$

Therefore $(P+Q)^{\dagger}G = P + (I-G)P + Q(I-G)$.

Corollary 2.3:

Let $P, Q \in B(X)$ be GD-Invertible such that $PQ - QP$ and $(I+P^{\dagger}Q)G$ is GD-Invertible in Minkowski space. If $PQ = QP = 0$, then $P^{\dagger}Q = Q^{\dagger}P = 0$ and $(P+Q)^{\dagger}G = P^{\dagger} + P^{\dagger}(I-G) + (I-G)Q^{\dagger}$.

Proof:

$$(P+Q)^{\dagger}G = P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I-QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(P^{\dagger})^{n+1}(-Q)^n\right]G + \left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I-PP^{\dagger}G)$$

$$P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G = P^{\dagger}(I+0)GQQ^{\dagger}G = P^{\dagger}GQQ^{\dagger}G.$$

Corollary 2.4:

If Q is quasi-Nilpotent, then $(P+Q)^{\dagger}G = P^{\dagger} + Q^{\dagger}(I-PP^{\dagger}G)$.

Proof:

Since Q is Quasi Nilpotent

$$P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G = 0$$

$$\begin{aligned}(I - QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G &= (I - QQ^{\dagger}G)[(-Q)^0(P^{\dagger})^{0+1} \\ &\quad +(-Q)^1(P^{\dagger})^{1+1} + \cdots]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger}QP^{\dagger^2} + Q^2P^{\dagger^3} - \cdots]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger}(I - QP^{\dagger} + Q^2P^{\dagger^2} - \cdots)]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger}(I + QP^{\dagger})^{\dagger}G \cdot G] \\ &= (I - 0)P^{\dagger}(I + 0) \\ &= P^{\dagger}\end{aligned}$$

and

$$\begin{aligned}\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I - PP^{\dagger}G) &= [(Q^{\dagger})^{0+1}(-P)^0 + (Q^{\dagger})^{1+1}(-P)^1 \\ &\quad +(Q^{\dagger})^{2+1}(-P)^2 + \cdots]G(I - PP^{\dagger}G) \\ &= [Q^{\dagger} - Q^{\dagger^2}P + Q^{\dagger^3}P^2 - \cdots]G(I - PP^{\dagger}G) \\ &= Q^{\dagger}(I - Q^{\dagger}P + Q^{\dagger^2}P^2 - \cdots)G(I - PP^{\dagger}G) \\ &= Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}G \cdot G \cdot (I - PP^{\dagger}G) \\ &= Q^{\dagger}(I + P^{\dagger}Q)(I - PP^{\dagger}G) \\ &= Q^{\dagger}(I - PP^{\dagger}G)\end{aligned}$$

Therefore $(P + Q)^{\dagger}G = P^{\dagger} + Q^{\dagger}(I - PP^{\dagger}G)$

Corollary 2.5:

If $Q^k = 0$, then

$$(P + Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I - QQ^{\dagger}G)\left[\sum_{n=0}^{k-1}(-Q)^n(P^{\dagger})^{n+1}\right]G + \left[\sum_{n=0}^{k-1}(Q^{\dagger})^{n+1}(-P)^n\right]G(I - PP^{\dagger}G)$$

Proof:

$$P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G$$

$$\begin{aligned}(I - QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G &= (I - QQ^{\dagger}G)[(-Q)^0(P^{\dagger})^{0+1} \\ &\quad +(-Q)^1(P^{\dagger})^{1+1} + \cdots]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger} - QP^{\dagger^2} + Q^2P^{\dagger^3} - \cdots]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger} - QP^{\dagger^2} + Q^2P^{\dagger^3} - \cdots 0]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger}(I + QP^{\dagger})^{\dagger}]G \cdot G \\ &= (I - QQ^{\dagger}G)[P^{\dagger}(I + QP^{\dagger})^{\dagger}]\end{aligned}$$

$$\begin{aligned}\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I - PP^{\dagger}G) &= [(Q^{\dagger})^{0+1}(-P)^0 + (Q^{\dagger})^{1+1}(-P)^1 + \\ &\quad (Q^{\dagger})^{2+1}(-P)^2 - \cdots]G(I - PP^{\dagger}Q) \\ &= [Q^{\dagger} - (Q^{\dagger})^2P + (Q^{\dagger})^3P^2 - \cdots]G(I - PP^{\dagger}G) \\ &= Q^{\dagger}[(I + Q^{\dagger}P)^{\dagger}](I - PP^{\dagger}G) \\ &= P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G(I - QQ^{\dagger})[P + (I + QP^{\dagger})^{\dagger}] \\ &\quad + Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}(I - PP^{\dagger}Q)\end{aligned}$$

Therefore $(P + Q)^{\dagger}G = (I + P^{\dagger}Q)^{\dagger}[P^{\dagger}GQQ^{\dagger}G + P^{\dagger}(I - QQ^{\dagger}G) + P^{\dagger}(I - QQ^{\dagger}G) + Q^{\dagger}(I - PP^{\dagger}G)]$

Corollary 2.6:

If $Q^2 = 0, (P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)GQQ^{\dagger}G + G +$

$$(I - QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G +$$

$$\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I - PP^{\dagger}G)$$

Proof:

$$P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G$$

$$\begin{aligned}(I - QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G &= (I - QQ^{\dagger}G)[(-Q)^0(P^{\dagger})^{0+1} \\ &\quad +(-Q)^1(P^{\dagger})^{1+1} + (-Q)^2(P^{\dagger})^{2+1} + \cdots]G \\ &= (I - QQ^{\dagger}G)[P^{\dagger} - Q(P^{\dagger})^2 + Q^2(P^{\dagger})^3 - \cdots]G \\ &= (I - QQ^{\dagger}G)P^{\dagger}(I - QP^{\dagger})\end{aligned}$$

$$\begin{aligned}\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I - PP^{\dagger}G) &= [(Q^{\dagger})^{0+1}(-P)^0 + (Q^{\dagger})^{1+1}(-P)^1 \\ &\quad +(Q^{\dagger})^{2+1}(-P)^2 - \cdots]G(I - PP^{\dagger}G) \\ &= [Q^{\dagger} - Q^{\dagger^2}P + Q^{\dagger^3}P^2 - \cdots]G(I - PP^{\dagger}G) \\ &= Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}G \cdot G \cdot (I - PP^{\dagger}G) \\ &= Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}(I - PP^{\dagger}G)\end{aligned}$$

Therefore $(P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I - QQ^{\dagger}G)P^{\dagger}(I - QP^{\dagger})G$
$+Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}(I - PP^{\dagger}G)$

Corollary 2.7:

If $Q^k = Q(k \geq 3)$, then $Q^{\dagger} = Q^{k-2}$

$$(P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G(I - P)^{\dagger}G\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G +$$

$$\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I - PP^{\dagger}G)$$

Proof:

$$\begin{aligned}P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G &= P^{\dagger}(I + Q^{\dagger}P)^{\dagger}GQ^{k-2}Q^{\dagger}G \\ &= P^{\dagger}(I + Q^{k-2}P)GQ^{k-2}Q^{\dagger}G \\ &= P^{\dagger}(I + Q^k - Q^{-2}P)^{\dagger}GQ^{k-2}Q^{\dagger}G \\ &= P^{\dagger}(I + Q \cdot Q^{-2}P)^{\dagger}GQ^{k-2}Q^{\dagger}G \\ &= P^{\dagger}(I + Q^{-1}P)^{\dagger}G \\ &= P^{\dagger}(I + Q^{\dagger}GP)^{\dagger}G\end{aligned}$$

$$(P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}GQ)$$

Corollary 2.8:

If $Q^2 = Q$. Then $Q^{\dagger} = Q$ and we have $(P+Q)^{\dagger}G = (P^{\dagger} + Q)GQG$
$+ (I - QG)[P^{\dagger} - P^{\dagger^2}Q(I + P)] + Q(I + P^{\dagger})(I - PP^{\dagger}G)$.

Proof:

$$
\begin{aligned}
P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G &= P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQ\cdot QG \qquad (\text{since } Q^{\dagger}=Q)\\
&= P^{\dagger}(I+Q^{\dagger}P)GQ^{\dagger}G\\
&= P^{\dagger}(I+QP)GQG \qquad (\text{since } Q^{2}=Q)\\
&= (P^{\dagger}+QPP^{\dagger})GQG\\
&= (P^{\dagger}+Q)GQG
\end{aligned}
$$

$$
\begin{aligned}
(I-QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(P^{\dagger})^{n+1}(-Q)^{n}\right]G &= (I-Q\cdot QG)[(P^{\dagger})^{0+1}(-Q)^{0}\\
&\quad +(P^{\dagger})^{1+1}(-Q)^{1}+(P^{\dagger})^{2+1}(-Q)^{2}+\cdots]G\\
&= (I-QG)[P^{\dagger}(I+P^{\dagger}Q)^{\dagger}G]G\\
&= (I-QG)[P^{\dagger}-P^{\dagger^{2}}Q+P^{\dagger^{3}}Q^{2}-P^{\dagger^{4}}Q^{3}+\cdots]G\\
&= (I-QG)[P^{\dagger}-P^{\dagger^{2}}Q+P^{\dagger^{3}}Q-P^{\dagger^{4}}Q^{2}+\cdots]G\\
&= (I-QG)P^{\dagger}[I-P^{\dagger}Q+P^{\dagger^{2}}Q-P^{\dagger^{3}}Q+\cdots]G\\
&= (I-QG)P^{\dagger}[I-P^{\dagger}Q(I-P^{\dagger}+P^{\dagger^{2}}-\cdots)]G\\
&= (I-QG)P^{\dagger}[I-P^{\dagger}Q(I+P^{\dagger})^{-1}G]\\
&= (I-QG)P^{\dagger}[I-P^{\dagger}Q(I+P^{\dagger})^{\dagger}G]G\\
&= (I-QG)[P^{\dagger}(I-P^{\dagger}Q(I+P))]\\
&= (I-QG)[P^{\dagger}-P^{\dagger^{2}}Q(I+P)]
\end{aligned}
$$

$$
\begin{aligned}
\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right]G(I-PP^{\dagger}G) &= \left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P^{n})\right]G(I-PP^{\dagger}G)\\
&= [Q^{0+1}(-P)^{0}+Q^{1+1}(-P)^{1}+Q^{2+1}(-P)^{2}+\cdots]G(I-PP^{\dagger}G)\\
&= [Q-Q^{2}P+Q^{3}P^{2}-\cdots]G(I-PP^{\dagger}G)\\
&= [Q-QP+QP^{2}-\cdots]G(I-PP^{\dagger}G) \qquad (\text{since } Q^{2}=Q^{\dagger}=Q)\\
&= [Q(I+P)^{-1}]G(I-PP^{\dagger}G)\\
&= Q(I+P)^{\dagger}G\cdot G(I-PP^{\dagger}G)\\
&= Q(I+P^{\dagger})(I-PP^{\dagger}G)
\end{aligned}
$$

Therefore

$$(P+Q)^{\dagger}G=(P^{\dagger}+Q)GQG+(I-QG)[P^{\dagger}-P^{\dagger^{2}}Q(I+P)]+Q(I+P^{\dagger})(I-PP^{\dagger}G).$$

Corollary 2.9:

If $P^{2}=P$ and $Q^{2}=Q$, then $I+PQ$ is invertible and $P(I+PQ)^{\dagger}GQ=\frac{1}{2}PQ$ and we have $(P+Q)^{\dagger}G=P^{\dagger}+Q^{\dagger}+(I-G)P^{\dagger}(I-P^{\dagger}Q-Q)+$

$$Q^{\dagger}(I-Q^{\dagger}P-P)(I-G).$$

$$
\begin{aligned}
&= [Q^\dagger - Q^{\dagger^2} P + Q^{\dagger^3} P - \cdots]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P + Q^{\dagger^2} P - \cdots]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P(I - Q^\dagger + Q^{\dagger^2} - \cdots)]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P(I + Q^\dagger)^\dagger G]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P(I+Q)](I-G) \\
&= Q^\dagger [I - Q^\dagger P - QQ^\dagger P](I-G) \\
&= Q^\dagger [I - Q^\dagger P - P](I-G)
\end{aligned}
$$

Therefore

$$(P+Q)^\dagger G = P^\dagger + Q^\dagger + (I-G)P^\dagger(I - P^\dagger Q - Q) + Q^\dagger(I - Q^\dagger P - P)(I-G).$$

Theorem 2.10:

Let $A \in B(X)$ and $B \in B(Y)$ are GD-invertible $C \in B(X,Y)$. Then $M = \begin{pmatrix} A & C \\ O & B \end{pmatrix}$ are GD-invertible and $M^\dagger = \begin{pmatrix} A^\dagger & X \\ O & B^\dagger \end{pmatrix}$, where

$$
\begin{aligned}
X &= (A^\dagger)^2 \left[\sum_{n=0}^{\infty} (A^\dagger)^n C B^n\right] G(I-G) \\
&\quad + (I-G)\left[\sum_{n=0}^{\infty} A^n C (B^\dagger)^n\right] G(B^\dagger)^2 - A^\dagger G C B^\dagger G.
\end{aligned}
$$

In [3], it is presented an expression of $(P+Q)^\dagger G$ under the condition that Q is quasi-nilpotent such that $P^\pi P Q = P^\pi Q P$ and $Q = QP^\pi$.

Proof:

$$
\begin{aligned}
P^\dagger(I + P^\dagger Q)^\dagger G Q Q^\dagger G &= P^\dagger(I + Q^\dagger P) G Q Q^\dagger G \\
&= (P^\dagger + Q^\dagger P P^\dagger) \cdot I \\
&= P^\dagger + Q^\dagger
\end{aligned}
$$

$$
\begin{aligned}
(I - QQ^\dagger G)\left[\sum_{n=0}^{\infty} (P^\dagger)^{n+1}(-Q)^n\right] G &= (I-G)[(P^\dagger)^{0+1}(-Q)^0 + \\
&\quad (P^\dagger)^{1+1}(-Q)^1 + (P^\dagger)^{2+1}(-Q)^2 + \cdots]G \\
&= (I-G)[P^\dagger - P^{\dagger^2} Q + P^{\dagger^3} Q^2 - \cdots]G \\
&= (I-G)[P^\dagger - P^{\dagger 2} Q + P^{\dagger^2} Q - \cdots]G \\
&= (I-G)P^\dagger[I - P^\dagger Q(I - P^\dagger + P^{\dagger^2} - \cdots)]G \\
&= (I-G)P^\dagger[I - P^\dagger Q(I + P^\dagger)^\dagger G \cdot G] \\
&= (I-G)P^\dagger[I - P^\dagger Q(I+P)] \\
&= (I-G)P^\dagger[I - P^\dagger Q - PQ^\dagger Q] \\
&= (I-G)P^\dagger(I - P^\dagger Q - Q)
\end{aligned}
$$

$$
\begin{aligned}
\left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right] G(I - PP^\dagger G) &= [(Q^\dagger)^{0+1}(-P)^0 + (Q^\dagger)^{1+1}(-P)^1 \\
&\quad + (Q^\dagger)^{2+1}(-P)^2 + \cdots]G(I-G)
\end{aligned}
$$

$$
\begin{aligned}
&= [Q^\dagger - Q^{\dagger^2} P + Q^{\dagger^3} P - \cdots]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P + Q^{\dagger^2} P - \cdots]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P(I - Q^\dagger + Q^{\dagger^2} - \cdots)]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P(I + Q^\dagger)^\dagger G]G(I-G) \\
&= Q^\dagger [I - Q^\dagger P(I+Q)](I-G) \\
&= Q^\dagger [I - Q^\dagger P - QQ^\dagger P](I-G) \\
&= Q^\dagger [I - Q^\dagger P - P](I-G)
\end{aligned}
$$

Therefore

$$(P+Q)^\dagger G = P^\dagger + Q^\dagger + (I-G)P^\dagger(I - P^\dagger Q - Q) + Q^\dagger(I - Q^\dagger P - P)(I-G).$$

Theorem 2.10:

Let $A \in B(X)$ and $B \in B(Y)$ are GD-invertible $C \in B(X,Y)$. Then $M = \begin{pmatrix} A & C \\ O & B \end{pmatrix}$ are GD-invertible and $M^\dagger = \begin{pmatrix} A^\dagger & X \\ O & B^\dagger \end{pmatrix}$, where

$$
\begin{aligned}
X &= (A^\dagger)^2 \left[\sum_{n=0}^{\infty} (A^\dagger)^n C B^n\right] G(I-G) \\
&\quad + (I-G)\left[\sum_{n=0}^{\infty} A^n C (B^\dagger)^n\right] G(B^\dagger)^2 - A^\dagger G C B^\dagger G.
\end{aligned}
$$

In [3], it is presented an expression of $(P+Q)^\dagger G$ under the condition that Q is quasi-nilpotent such that $P^\pi PQ = P^\pi QP$ and $Q = QP^\pi$.

In fact, if we answer that $P^\pi Q(1-P^\pi) = 0$, instead of Q quasi-nilpotent with $Q = QP^\pi$, we will get a general result.

Theorem 2.11:

Let $P \in B(X)$ be GD-invertible in Minkowski space $\mathscr{M}$ and $Q \in B(X)$ such that $||QP^\dagger|| < 1$, $P^\pi Q(1-P^\pi) = 0$ and $P^\pi PQ = P^\pi Q$ is GD-invertible in m, then $(P+Q)$ is GD-invertible. In this case,

$$
\begin{aligned}
(P+Q)^\dagger G &= (I+P^\dagger Q)^\dagger G P^\dagger G + (I+P^\dagger Q)^\dagger G(I - PP^\dagger G)\left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right] G \\
&\quad + \left[\sum_{n=0}^{\infty} ((I+P^\dagger Q)^\dagger G P^\dagger)^{n+2} Q(I - PP^\dagger G)(P+Q)^n\right](I - PP^\dagger G) \\
&\quad \times \left\{ I - (P+Q)(I - PP^\dagger G)\left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right] G \right\}.
\end{aligned}
$$

Proof:

Since P is GD-invertible and $P^\pi Q(1-P^\pi) = 0, P$ and Q have the form

$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ and $Q = \begin{pmatrix} Q_1 & Q_3 \\ 0 & Q_2 \end{pmatrix}$ with respect to the space decomposition $X = N(P^\pi) \oplus R(P^\pi)$, where P_1 is invertible and P_2 is quasi-nilpotent.

$||QP^\dagger G|| < 1$ implies that $I + P^\dagger Q$ is invertible.

$P^\pi QP = P^\dagger PQ$ implies that $P_2Q_2 = Q_2P_2$. Since $P^\pi Q$ is GD-invertible, Q_2 is GD-invertible. It follows from Theorem 2.1 that

$(P_2 + Q_2)^\dagger G = \left[\sum_{n=0}^{\infty} (Q_2^\dagger)^{n+2}(-P_2)^2\right] G$

By Theorem 2.10, $(P + Q)^\dagger G$ has the form

$$(P+Q)^\dagger G = \begin{bmatrix} (P_1+Q_1)^\dagger G & S \\ 0 & \sum_{n=0}^{\infty} (Q_2^\dagger)^{n+2}(-P_2)^n \end{bmatrix},$$

$$\text{where } S = \left[\sum_{n=0}^{\infty} ((P_1+Q_1)^\dagger)^{n+2} GQ_3(P_2+Q_2)^n\right]$$
$$\left[I - (P_2+Q_2)\left(\sum_{n=0}^{\infty} (Q_2^\dagger)^{n+1}(-P_2)^n\right) G\right]$$
$$-(P_1+Q_1)^\dagger GQ_3\left(\sum_{n=0}^{\infty} (Q_2^\dagger)^{n+1}(-P_2)^n\right) G.$$

Note that the product and the sum pf $P, Q, P^\dagger G$ and $Q^\dagger G$ are still the upper triangular operator matrices.

Thus $(P_1 + Q_1)^\dagger G = P_1^\dagger (I + P_1^\dagger Q)^\dagger G$

and

$$0 \oplus \left(\sum_{n=0}^{\infty} (Q_2^\dagger)^{n+1}(-P_2)^n\right) G = P^\pi \left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right] G$$

$$\begin{bmatrix} (P_1+Q_1)G & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (P_1+Q_1)G & 0 \\ 0 & \left(\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right) G \end{bmatrix} + \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

$$= P^\dagger(I+P^\dagger Q)^\dagger G + P^\dagger(I+P^\dagger Q)^\dagger G P^\pi \left[\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right] G +$$
$$\left[\sum_{n=0}^{\infty} ((I+P^\dagger Q)^\dagger P^\dagger)^{n+2} GQP^\pi (P+Q)^n\right] P^\pi$$
$$\left[I - (P+Q)P^\pi \left(\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right) G\right]$$
$$- P^\dagger(I+P^\dagger Q)^\dagger G \cdot GP^\dagger QP^\pi \left(\sum_{n=0}^{\infty} (Q^\dagger)^{n+1}(-P)^n\right) G$$

$$= -P^{\dagger}(I + P^{\dagger}Q)^{\dagger}G \cdot GP^{\dagger}QP^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G$$

$$+ (I + P^{\dagger}Q)^{\dagger}P^{\dagger}QP^{\pi}(I + Q^{\dagger}P)^{\dagger}G \cdot G \qquad [\because P^{\dagger}Q = 0]$$

$$(P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}G + P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GP^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G$$

$$+ \left[\sum_{n=0}^{\infty}((I + P^{\dagger}Q)^{\dagger}P^{\dagger})^{n+2}GQP^{\pi}(P+Q)^{n}\right]P^{\pi}$$

$$\left[I - (P+Q)P^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G\right].$$

Corollary 2.12:

Let $P \in B(X)$ be GD-invertible and $Q \in B(X)$ such that $||QP^{\dagger}G|| < 1$, $P^{\pi}Q(1 - P^{\pi}) = 0$ and $P^{\pi}PQ = P^{\pi}QP$.

1. If $QPP^{\dagger}G = 0$ and Q is quasi-nilpotent, then Theorem 2.11 is simplified to

$$(P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}G + P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GP^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G$$

$$+ \left[\sum_{n=0}^{\infty}((I + P^{\dagger}Q)^{\dagger}P^{\dagger})^{n+2}GQP^{\pi}(P+Q)^{n}\right]$$

$$P^{\pi}\left[I - (P+Q)P^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G\right]$$

$$P^{\pi}\left[I - (P+Q)P^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G\right]$$

$$P^{\dagger}(I + P^{\dagger}Q)^{\dagger}G = P^{\dagger}(I + 0)^{\dagger}G \qquad \text{(since } Q \text{ is quasi-nilpotent)}$$

$$= P^{\dagger}G$$

$$P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GP^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G = P^{\dagger}(I+0)^{\dagger}GP^{\pi}G^{\dagger}(I+Q^{\dagger}P)^{\dagger}G \cdot G$$

$$= P^{\dagger}GP^{\pi}Q^{\dagger}(I + P^{\dagger}Q)$$

$$= P^{\dagger}GP^{\pi}(Q^{\dagger} + P^{\dagger})$$

$$\left[\sum_{n=0}^{\infty}((I + P^{\dagger}Q)^{\dagger}P^{\dagger})^{n+2}GQP^{\pi}(P+Q)^{n}\right]$$

$$P^{\pi}\left[I - (P+Q)P^{\pi}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G\right]$$

$$= \left[\sum_{n=0}^{\infty}(P^{\dagger})^{n+2}GQ(P+Q)^{n}\right](I-0)[I - PQ^{\dagger}(I + Q^{\dagger}P)^{\dagger}G \cdot G]$$

$$\text{(since } QP^{\pi} = Q)$$

$$\left[\sum_{n=0}^{\infty}(P^{\dagger})^{n+2}GQ(P+Q)^{n}\right][I - PQ^{\dagger}(I + P^{\dagger}Q)]$$

$$= \left[\sum_{n=0}^{\infty} (P^{\dagger})^{n+2} GQ(P+Q)^n \right] [I - PQ^{\dagger}(I)]$$

$$= \left[\sum_{n=0}^{\infty} (P^{\dagger})^{n+2} GQ(P+Q)^n \right] [I - PQ^{\dagger}]$$

$$\therefore (P+Q)^{\dagger} G = P^{\dagger} G + P^{\dagger} G P^{\pi} (Q^{\dagger} + P^{\dagger}) + \left[\sum_{n=0}^{\infty} (P^{\dagger})^{n+2} GQ(P+Q)^n \right] (I - PG^{\dagger})$$

2. If $P^{\pi} Q = QP^{\pi}, \sigma(P^{\pi} Q) = 0,$

$$(P+Q)^{\dagger} G = P^{\dagger} (I + P^{\dagger} Q)^{\dagger} G$$

$$(I + P^{\dagger} Q)^{\dagger} G P^{\pi} \left(\sum_{n=0}^{\infty} (Q^{\dagger})^{n+1} (-P)^n \right) G = 0$$

$$\left[\sum_{n=0}^{\infty} ((I + P^{\dagger} Q)^{\dagger} P^{\dagger})^{n+2} GQP^{\pi} (P6Q)^n \right]$$

$$P^{\pi} \left[I - (P+Q) P^{\pi} \left(\sum_{n=0}^{\infty} (Q^{\dagger})^{n+2} (-P)^n \right) G \right] = 0$$

Therefore

$$(P+Q)^{\dagger} G = P^{\dagger} (I + P^{\dagger} Q)^{\dagger} G.$$

Theorem 2.13:

Let P and $Q \in B(X)$ be GD-invertible. Let F be an idempotent such that $FP = PF, (I-F)QF = 0, (PQ - QP)F = 0$ and $(I-F)(PQ-QP) = 0$. If $(P+Q)F$ and $(I-F)(P+Q)$ are GD-invertible, than $(P+Q)$ is GD-invertible and

$$(P+Q)^{\dagger} G = \sum_{n=0}^{\infty} \Delta^{n+2} FQ(I-F)(P+Q)^n FQ(I-F)\Delta^{n+2} + (I - \Delta FQ)(I-F)\Delta + \Delta F,$$

where

$$\Delta = P^{\dagger} (I + P^{\dagger} Q)^{\dagger} GQQ^{\dagger} G + (I - QQ^{\dagger} G) \left(\sum_{n=0}^{\infty} (-Q)^n (P^{\dagger})^{n+1} \right) G + \left(\sum_{n=0}^{\infty} (Q^{\dagger})^{n+1} (-P)^n \right) G (I - PP^{\dagger} G) \quad (1.9)$$

Proof:

Since $F^2 = F$, we have $F = I \oplus 0$ with respect to space decomposition $X = R(F) \oplus N(F)$. From $FP = PF$ and $(I-F)QF = 0$, we know that

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}; Q = \begin{pmatrix} Q_1 & Q_3 \\ 0 & Q_2 \end{pmatrix} \quad (1.10)$$

Since $(P+Q)F$ and $(I-F)(P+Q)$ are GD-invertible $P_i + Q_i (i = 1, 2)$ are GD-invertible.

Hence $(P+Q)^{\dagger}G = \begin{pmatrix} (P_1+Q_1)^{\dagger}G & X \\ 0 & (P_2+Q_2)^{\dagger}Q \end{pmatrix}$

where

$$X = \left[\sum_{n=0}^{\infty}((P_1+Q_1)^{\dagger}G)^{n+2}Q_3(P_2+Q_2)^n\right][I-(P_2+Q_2)(P_2+Q_2)^{\dagger}G]$$
$$+[I-(P_1+Q_1)(P_1+Q_1)^{\dagger}G]\left[\sum_{n=0}^{\infty}(P_1+Q_1)^nQ_3((P_2+Q_2)^{\dagger}G)^{n+2}\right]$$
$$-((P_1+Q_1)^{\dagger}G)Q_3(P_2+Q_2)^{\dagger}G.$$

By Theorem 2.10, from $(PQ-QP)F=0$ and $(I-F)(PQ-QP)=0$.

We know that $P_iQ_i = Q_iP_i$ $(i=1,2)$, note that $P, Q, P^{\dagger}G$ and $A^{\dagger}G$ are all the upper triangular operator matrices by Theorem 2.1. It shows that

$$\begin{aligned}(P_1+Q_1)^{\dagger}G\oplus 0 = \; & P_1^{\dagger}(I+P_1^{\dagger}Q_1)^{\dagger}GQ_1Q_1^{\dagger}G\oplus 0 + \\ & (I-Q_1Q_1^{\dagger}G)\left(\sum_{n=0}^{\infty}(-Q_1^{\dagger})^n(P_1^{\dagger})^{n+1}\right)G\oplus 0 + \\ & \left(\sum_{n=0}^{\infty}(Q_1^{\dagger})^{n+1}(-P_1)^n\right)(I-P_1P_1^{\dagger}G)\oplus 0 \\ = \; & P^{\dagger}(IP^{\dagger}Q)^{\dagger}GQQ^{\dagger}GF + \\ & (I-QQ^{\dagger}G)\left(\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right)GF + \\ & \left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right)G(I-PP^{\dagger}G)F\end{aligned}$$

$$(P_1+Q_1)^{\dagger}G\oplus 0 = \Delta F \tag{1.11}$$

Where Δ is defined in Equation (1.9).

Similarly we can prove

$$\begin{aligned}0\oplus(P_2+Q_2)^{\dagger}G = \; & 0\oplus P_2^{\dagger}(I+P_2^{\dagger}Q_2)^{\dagger}GQ_2Q_2^{\dagger}(I-Q_2Q_2^{\dagger}G) \\ & \left(\sum_{n=0}^{\infty}(-Q_2)^n(P_2\dagger)^{n+1}\right)G + \\ & \left(\sum_{n=0}^{\infty}(Q_2^{\dagger})^{n+1}(-P_2)^n\right)G(IP_2P_2^{\dagger}G)\end{aligned}$$

$$0\oplus P_2^{\dagger}(I+P_2^{\dagger}Q_2)^{\dagger} = (I-F)P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G \qquad (\text{since } F = I\oplus 0]$$

$$0\oplus(I-Q_2Q_2^{\dagger}G)\left(\sum_{n=0}^{\infty}(-Q_2)^n(P_2^{\dagger})^{n+1}\right)G = (I-F)(I-QQ^{\dagger}G)$$

$$\left(\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right)G$$

$$0\oplus\left(\sum_{n=0}^{\infty}(Q_2^{\dagger})^{n+1}(-P_2)^n\right)G(I-P_2P_2^{\dagger}G)=(I-F)\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right)$$

$$G(I-PP^{\dagger}G)$$

$$0\oplus(P_2+Q_2)^{\dagger}G=(I-F)P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G+(I-F)(I-QQ^{\dagger}G)$$

$$\left(\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right)G+(I-F)\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P^{\dagger})^n\right)G(I-PP^{\dagger}G)$$

$$=(I-F)\left\{P^{\dagger}(I+P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G+(I-QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]G\right.$$

$$\left.+\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right)G(I-PP^{\dagger}G)\right\}$$

$$=(I-F)\Delta \tag{1.12}$$

We observe that,

$$\begin{pmatrix}0 & ((P_1+Q_1)^{\dagger}G)^{n+2}Q_3(P_2+Q_2)^n(I-(P_2+Q_2)(P_2+Q_2)+G)\\ 0 & 0\end{pmatrix}$$

$$[\because (P_1+Q_1)^{\dagger}G=\Delta F]$$

$$=[(\Delta F)^{n+2}FQ(I-F)(P+Q)^n][I-(P+Q)(I-F)\Delta]$$

$$=[\Delta^{n+2}FQ(I-F)(P+Q)^n][I-(P+Q)\Delta]$$

(since F is idempotent]

with

$$\begin{pmatrix}0 & [I-(P_1+Q_1)(P_1+Q_1)^{\dagger}G][(P_1+Q_1)^nQ_3((P_2+Q_2)^{\dagger}G)^{n+2}]\\ 0 & 0\end{pmatrix}$$

$$=[I-(P+Q)\Delta F][(P+Q)^nFQ(I-F)((I-F)\Delta)^{n+2}]$$

(since F is idempotent)

$$=(I-(P+Q)\Delta)(P+Q)^nF^2Q(I-F)\Delta^{n+2}$$

$$=(I-(P+Q)\Delta)(P+Q)^nFQ(I-F)\Delta^{n+2}$$

and

$$\begin{pmatrix} 0 & -(P_1+Q_1)+GQ_3(P_2+Q_2)+G \\ 0 & (P_2+Q_2+G) \end{pmatrix} = \begin{pmatrix} 0 & -(\Delta F)FQ(I-F)\Delta \\ 0 & (I-F)\Delta \end{pmatrix}$$
$$= -\Delta F^2 Q(I-F)\Delta + (I-F)\Delta$$
$$\text{(since } F \text{ is idempotent)}$$
$$= (I-F)\Delta[I-\Delta FQ]$$

Hence

$$(P+Q)^\dagger = \Delta^{n+2}FQ(I-F)(P+Q)^n[I-(P+Q)\Delta]+[I-(P+Q)\Delta](P+Q)^n FQ$$
$$(I-F)\Delta^{n+2} + (I-F)\Delta(I-\Delta FQ) + \Delta F$$

Therefore

$$(P+Q)^\dagger G = \sum_{n=0}^{\infty} \Delta^{n+2}FQ(I-F)(P+Q)^n(I-(P+Q)\Delta)+$$
$$[I-(P+Q)\Delta](P+Q)^n FQ(I-F)\Delta^{n+2}+$$
$$(I-F)\Delta(I-\Delta FQ)+\Delta F.$$

Hence Proved.

III. Special Cases

Let us use Theorem 2.13 to analyze some interesting special perturbations of linear operators. We thereby extend earlier work by several authors [4, 6, 7, 12, 13] and partially solve a problem posed in 1975 by Campbell and Meyer, who consider it difficult to establish the norm estimates for the perturbation of the Drazin inverse.

Case (i): If $QF = 0$, then $\Delta F = P^\dagger GF, Q\Delta F = 0$, thus Theorem 2.13 reduces to

$$(P+Q)^\dagger G = \sum_{n=0}^{\infty} \Delta^{n+2}FQ(I-F)(P+Q)^n(I-F)[I-(P+Q)\Delta]+[I-(P+Q)\Delta]F$$
$$\sum_{n=0}^{\infty}(P+Q)^n FQ(I-F)\Delta^{n+2} + (I-\Delta FQ)(I-F)\Delta + \Delta F$$

$$(P+Q)^\dagger G = \sum_{n=0}^{\infty}(P+Q)^{n+2}FQ(P+Q)^n[I-(P+Q)\Delta] + (I-PP^\dagger G)$$

$$\sum_{n=0}^{\infty}(P+Q)^n FQ\Delta^{n+2} - P^{\dagger}GFQ\Delta + (I-F)\Delta + P^{\dagger}GF$$

$$(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}\Delta^{n+2}FQ(I-F)(P+Q)^n(I-F)[I-(P+Q)\Delta]+[I-(P+Q)\Delta]F$$

$$\sum_{n=0}^{\infty}(P+Q)^n FQ(I-F)\Delta^{n+2} + (I-\Delta FQ)(I-F)\Delta + \Delta F$$

$$\Delta F = 0, \Delta F = P^{\dagger}GF, \Delta = P^{\dagger}G, Q\Delta F = 0$$

$$(P+G)^{\dagger}G = \sum_{n=0}^{\infty} FQ(P+Q)^n(I-(P+Q)\Delta)$$

$$[I-(P+Q)\Delta] = \sum_{n=0}^{\infty}(P+Q)^n FQ(I-F)\Delta^{n+2} = (I-(P\Delta F + Q\Delta F))$$

$$\sum_{n=0}^{\infty}(P+Q)^n FQ\Delta^{n+2}$$

$$= (I-PP^{\dagger}G)\sum_{n=0}^{\infty}(P+Q)^n FQ\Delta^{n+2}$$

$$(I-\Delta FQ)(I-F)\Delta + \Delta F = (\Delta - \Delta FQ\Delta)(I-F)$$

$$= \Delta - \Delta FQ\Delta - \Delta F + \Delta FQ\Delta F$$

$$= \Delta - \Delta FQ\Delta - \Delta F + O$$

$$= -\Delta FQ\Delta + \Delta - \Delta F$$

$$= P^{\dagger}GFQ\Delta + (I-F)\Delta + \Delta F$$

$$= P^{\dagger}GFQ\Delta + (I-F)\Delta + P^{\dagger}GF$$

Therefore $(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}(P^{\dagger}G)^{n+2}FQ(P+Q)^n(I-(P+Q)\Delta)+[I-PP^{\dagger}G]$

$$\sum_{n=0}^{\infty}(P+Q)^n FQ\Delta^{n+2} - P^{\dagger}GFQ\Delta + (I-F)\Delta + P^{\dagger}GF.$$

Case (i)a:

If $QF = 0$ and $F = I - PP^{\dagger}G$, then $P^{\dagger}GF = 0, (P+Q)^n = P^n F$

Case (i) is $(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}(P^{\dagger}G)^{n+2}FQ(P+Q)^n(I-(P+Q)\Delta)+(I-PP^{\dagger}G)$

$$\sum_{n=0}^{\infty}(P+G)^n FQ\Delta^{n+2} - P^{\dagger}GFQ\Delta + (I-F)\Delta + P^{\dagger}GF$$

$$\sum_{n=0}^{\infty}(P^{\dagger}G)^{n+2}FQ(P+Q)^n = \sum_{n=0}^{\infty}(P^{\dagger}G)^n(P^{\dagger}G)^2 FQ(P+Q)^n$$

$$= 0$$

$$(I - PP^{\dagger}G) \sum_{n=0}^{\infty} (P+Q)^n FQ\Delta^{n+2} = F \sum_{n=0}^{\infty} P^n FQ\Delta^{n+2}$$

$$= \sum_{n=0}^{\infty} P^n F^2 Q\Delta^{n+2}$$

$$= \sum_{n=0}^{\infty} P^n FQ\Delta^{n+2}$$

$$= \sum_{n=0}^{\infty} P^n (I - PP^{\dagger}G) Q\Delta^{n+2}$$

$$-P^{\dagger}GFQ\Delta + (I-F)\Delta + P^{\dagger}GF = 0 + (I-F)\Delta + 0$$

$$= PP^{\dagger}G\Delta$$

Therefore $(P+Q)^{\dagger}G = \sum_{n=0}^{\infty} P^n (I - PP^{\dagger}G) G\Delta^{n+2} + PP^{\dagger}G\Delta.$

Case (i).a.1:

If $QF = 0, F = I - PP^{\dagger}G$ and Q is quasi-nilpotent. Then by Corollary 2(3).

If Q is quasi-nilpotent.

Then $(P+Q)^{\dagger}G = \sum_{n=0}^{\infty} (P^{\dagger}G)^{n+1}(-Q)^n = (I + P^{\dagger}Q)^{\dagger}GP^{\dagger}G$

Case (i).a.1:

If $QF = 0, F = I - PP^{\dagger}G$ and Q is quasi-nilpotent. Then by Corollary 2(3).

If Q is quasi-nilpotent.

Then $(P+Q)^{\dagger}G = \sum_{n=0}^{\infty} (P^{\dagger}G)^{n+1}(-Q)^n = (I + P^{\dagger}Q)^{\dagger}GP^{\dagger}G$

$$(P+Q)^{\dagger}G = P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G + (I - QQ^{\dagger}G)\left(\sum_{n=0}^{\infty}(-Q)^n (P^{\dagger})^{n+1}\right)G +$$

$$\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P^{\dagger})^n\right)G(I - PP^{\dagger}G)$$

$$P^{\dagger}(I + P^{\dagger}Q)^{\dagger}GQQ^{\dagger}G = 0$$

$$(I - QQ^{\dagger}G)\left(\sum_{n=0}^{\infty}(-Q)^n (P^{\dagger})^{n+1}\right)G = 0$$

$$\left(\sum_{n=0}^{\infty}(-Q)^n (P^{\dagger})^{n+1}\right)G(I - PP^{\dagger}G) = \left(\sum_{n=0}^{\infty}(-Q)^n (P^{\dagger})^{n+1}\right)G(I - PP^{\dagger}G)$$

$$= \left(\sum_{n=0}^{\infty}(-Q)^n (P^{\dagger})^{n+1}\right)GF$$

$$= [(Q^{\dagger})^{0+1}(-P)^0 + (Q^{\dagger})^{1+1}(-P)^1 + \cdots]GF$$

$$= Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}G \cdot GF$$

$$= Q^{\dagger}(I + Q^{\dagger}P)^{\dagger}F$$

$PP^{\dagger}G\Delta = PP^{\dagger}GQ^{\dagger}(I+Q^{\dagger}P)^{\dagger}F$

Therefore Case (1.a) becomes,

$(P+Q)^{\dagger}G = \sum_{n=0}^{\infty} P^n(I-PP^{\dagger}G)Q\Delta^{n+2} + PP^{\dagger}GQ^{\dagger}(I+Q+P)^{\dagger}F.$

Case (ii)a.2:

If $QF = FQ = 0, F = I - PP^{\dagger}G$ and Q is quasi-nilpotent, then Case (1.a.1) turns into

$$(P+Q)^{\dagger}G = P^{\dagger}(I+P+Q)^{\dagger}GQQ^{\dagger}G + (I-QQ^{\dagger}G)\left[\sum_{n=0}^{\infty}(-Q)^n(P^{\dagger})^{n+1}\right]$$
$$G + \left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I-PP^{\dagger}G)$$
$$= 0 + 0\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]G(I-PP^{\dagger}G)$$
$$= Q^{\dagger}(I+Q^{\dagger}P)^{\dagger}G\cdot G(I-PP^{\dagger}G)$$
$$= Q^{\dagger}(I+Q^{\dagger}P)^{\dagger}F$$

Therefore $(P+Q)^{\dagger}G = Q^{\dagger}(I+Q^{\dagger}P)^{\dagger}F.$

Case (i)b:

If $QF = 0$ and $F = PP^{\dagger}G$, then $(P+Q)^nF = P^nF = FP^nF = F(P+Q)^nF$ So we have $(I-PP^{\dagger})\sum_{n=0}^{\infty}(P+Q)^nFQ\Delta^{n+2} = 0$

Since $-P^{\dagger}FQ\Delta + (I-F)\Delta + P^{\dagger}GF = (I-P^{\dagger}Q)(I-F)\Delta + P^{\dagger}GF$

$= (I+P^{\dagger}Q) - 1P^{\pi}\Delta + P^{\pi}F$ and

$P^{\pi}\Delta = P^{\pi}\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n$ by Corollary 2(3) and Equation (10), Case (1) becomes

$$(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}(P^{\dagger})^{n+1}Q(P+Q)^n\left[I-(P+Q)\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n\right]$$
$$+(I+P^{\dagger}Q)^{-1}P^{\pi}\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^n + P^{\dagger}.$$

Case (i)b.1:

If $QF = 0, F = PP^{\dagger}G$ and Q is quasi-nilpotent then Case (1.b) can be simplified as

$$(P+Q)^{\dagger}G=\left[\sum_{n=0}^{\infty}(P^{\dagger})^{n+2}Q(P+Q)^{n}\right]G\left[I-(P+Q)\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right]G$$

$$+(I+P^{\dagger}Q)^{\dagger}GP^{\dagger}\left(\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right)G+P^{\dagger}G$$

$$=0+(I+P^{\dagger}Q)^{\dagger}G(I-PP^{\dagger}G)\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right]G+P^{\dagger}G$$

$$=0+(I+0)^{\dagger}G(I-F)\left[\sum_{n=0}^{\infty}(Q^{\dagger})^{n+1}(-P)^{n}\right]G+P^{\dagger}G$$

$$=G(I-F)Q^{\dagger}(I+Q^{\dagger}P)^{\dagger}GG+P^{\dagger}G$$

$$(P+Q)^{\dagger}G=G(I-F)Q^{\dagger}(I+Q^{\dagger}P)^{\dagger}+P^{\dagger}G.$$

Case (i).b.2

If $QF=FQ=0, F=PP^{\dagger}G$ and Q is quasi-nilpotent then Case (1.b.1) becomes

$$(P+Q)^{\dagger}G=G(I-F)Q^{\dagger}(I+Q^{\dagger}P)^{\dagger}+P^{\dagger}G$$

$$=G(I-F)Q^{\dagger}(I+P^{\dagger}Q)+P^{\dagger}G$$

$$=G(I-F)Q^{\dagger}(I+0)+P^{\dagger}G$$

$$(P+Q)^{\dagger}G=G(I-F)Q^{\dagger}+P^{\dagger}G.$$

Case (ii):

If $QF=(I-F)Q=0$, then $\Delta=P^{\dagger}G-P^{\dagger}GQP^{\dagger}G, (I-F)(P+Q)^{n}=(I-F)P^{n}$

$(P+Q)^{n}F=P^{n}F, (I-F)[I-(P+Q)0]=(I-F)(I-PP^{\dagger}G)$ and $(I-F)\Delta=(I-P)P^{\dagger}G.$

Theorem 2.13 reduces to

$$(P+Q)^{\dagger}G=\sum_{n=0}^{\infty}\Delta^{n+2}FQ(I-F)(P+Q)^{n}[I-(P+Q)\Delta]+[I-(P+Q)\Delta]$$

$$(P+Q)^{n}FQ(I-F)\Delta^{n+2}+(I-F)\Delta(I-\Delta FQ)+\Delta F$$

$$\sum_{n=0}^{\infty}\Delta^{n+2}FQ(I-F)(P+Q)^{n}[I-(P+Q)\Delta]=\sum_{n=0}^{\infty}(P^{\dagger})^{n+2}QP^{n}(I-PP^{\dagger}G)$$

$$[I-(P+Q)\Delta](P+Q)^{n}FQ(I-F)\Delta^{n+2}=(I-PP^{\dagger}G)P^{n}Q(P^{\dagger})^{n+2}.$$

Since $(I-F)(P+Q)^n = (I-F)P^n$

$$(I-F)\Delta(I-\Delta FQ)+\Delta F = P^{\dagger}G + P^{\dagger}G - (P^{\dagger}QP^{\dagger})G$$

$$= 2P^{\dagger}G - P^{\dagger}GQP^{\dagger}G$$

Therefore

$$(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}(P^{\dagger})^{n+2}QP^n(I-PP^{\dagger}G) + (I-PP^{\dagger}G)\sum_{n=0}^{\infty}P^nQ(P^{\dagger})^{n+2}$$
$$+2P^{\dagger}G - P^{\dagger}GQP^{\dagger}G.$$

Case (ii)a:

If $QF = (I-F)Q = P(I-F) = 0$, then $QP = QP^{\dagger}G = 0$. Then Case (ii) is

$$(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}(P^{\dagger})^{n+2}QP^n(I-PP^{\dagger}G) + (I-PP^{\dagger}G)\sum_{n=0}^{\infty}P^nQ(P^{\dagger})^{n+2}$$
$$+2P^{\dagger}G - P^{\dagger}GQP^{\dagger}Q$$
$$= (P^{\dagger})^2Q(I-PP^{\dagger}G) + (I-PP^{\dagger}G)Q(P^{\dagger})^2 + 2P + G - 0$$
$$= (P^{\dagger})^2(Q-QPP^{\dagger}G) + (I-PP^{\dagger}G)0 + 2P^{\dagger}G$$
$$= (P^{\dagger})^2Q + 2P^{\dagger}G.$$

Case (ii)b:

If $QF = (I-F)Q = FP = 0$, then $PQ = P^{\dagger}GQ = 0$. Then Case (ii) turns

$$(P+Q)^{\dagger}G = \sum_{n=0}^{\infty}(P^{\dagger})^{n+2}QP^n(I-PP^{\dagger}G) + (I-PP^{\dagger}G)\sum_{n=0}^{\infty}P^nQ(P^{\dagger})^{n+2}$$
$$+2P^{\dagger}G - P^{\dagger}GQP^{\dagger}G$$
$$= (P^{\dagger})^2a(I-PP^{\dagger}G) + (I-PP^{\dagger}G)(1)Q(P^{\dagger})^2$$
$$+2P^{\dagger}G - 0$$
$$= 0 + (I-PP^{\dagger}G)2P^{\dagger^2} + 2P^{\dagger}G$$

$$(P+Q)^{\dagger}G = Q(P^{\dagger})^2 + 2P^{\dagger}G.$$

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