IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 5 Ver. VIII (Sep. - Oct.2016), PP 08-20
www.iosrjournals.org

Shannon Wavelet Analysis with Applications: A Survey

Zaheer Abbas[1] Princess Raina[2]
[1,2] *Department of Mathematical Sciences, BGSB University Rajouri*

Abstract: *Among the many families of wavelets available in the literature, Shannon wavelets offer some more specific advantages which are usually missing in the others such as infinite differentiability, analytical expressions, shapely boundedness in the frequency domain, enjoying a generalization of the Shannon sampling theorem, giving rise to the connection coefficients which can be analytically defined for any order derivatives and since it is a known fact that wavelets have been finding enormous applications in science and technology since 1980, so it becomes very fruitful to study the Shannon wavelet and its applications. In the literature a lot of study have been done in this direction, but to our knowledge no comprehensive survey of Shannon wavelet analysis and its applications have been done and hence we take this opportunity. The paper begins with a brief historical Journey from Fourier analysis to wavelet analysis presented in first section. Afterwards, Shannon wavelet, Shannon scaling function, reconstruction of functions and derivatives with their help have been presented in the next four sections. In next eleven sections, applications of Shannon wavelet analysis to differential, integral and integro-differential equations, inverse heat conduction problem, and some real world problems have been surveyed. Finally conclusion is included in the last section.*

Keywords and phrases: *Connection coefficients, C^2- functions, Integro – differential equations, Multiresolution Analysis, Shannon scaling function, Shannon wavelet.*

I. Journey From Fourier To Wavelet Analysis

To study matter, the process becomes simpler if we start understanding quarks, then atoms, then molecules and finally matter. Similarly properties of integers can be more understood if we start working with primes and then proceed to analyse integers. Similarly one can understand the organisms if one firstly understands cells. As a general rule, "the process of understanding becomes easier if complicated structures are synthesized by using simpler ones". Jean Baptiste Joseph Fourier did the same thing. In early 1800's the most important problem that the scientists were facing was "How heat diffuses in a continuous medium". Along with others, Jean Baptiste Joseph Fourier was also trying to solve the problem. During his attempt of putting forward a solution, an idea(which later on proved to be one of the most important ideas in the history of science and technology) of synthesizing a function with the help of simpler functions – Sines and Cosines –came in his mind. As a result, on December 21,1807, Fourier submitted a manuscript to "institute de France" in which he claimed that "every periodic function can be expressed as a weighted sum of sines and cosines". His manuscript went to a committee consisting of Laplace, Langrange, Lacroix and Monge (At that time Poisson was only the committee's clerk). As the Fourier's work was based on intuitions and no rigorous approach was followed, in particular with respect to the convergence of the series, the committee rejected Fourier's paper for publication(actually, Langrange opposed most strongly among the members of the committee). Thus, finally in 1808 Poisson (clerk of the committee at that time) put forward the committee's report which was: "Rejected on the grounds that it contained nothing new or interesting". This rejection did not make big set back to the Fourier's mission, probably he knew the tradition of this world: "Whenever somebody comes with a new idea ,people ,especially prominent ones ,oppose it".

He once again came, with few modifications, particularly in rigrouring his idea, in 1811, essentially to the same committee ,as `a candidate for the "Grand pring de mathematiques" for 1812. This time again the committee did not allow his paper for publication in "Memories de e'Academie des sciences", although they awarded him the prize. In 1824, things changed and Fourier became the secretary of the Academie and then he published his work, without practically changing his older version.

Although Fourier's idea later revolutionized the science and technology, but the problem of convergence highlighted by Lagrange was really serious. Thus some famous brains of that time - Poisson, Cauchy, Drichlet and many others got engaged in resolving this problem. In 1826 Cauchy published a proof of convergence of Fourier series, but there was a flaw in his proof which, three years later, a 23-years old boy Drichlet, highlighted. He (Drichlet) himself gave some sufficient conditions for the convergence of Fourier series (Drichlet point wise convergence theorem), which later in 1881 were improved by Jorden. In the last century, there had been many new sufficient conditions for the convergence of Fourier series put forward by many mathematicians[1].

Because of the introduction of the Fourier series many new concepts either popped up or were made more rigorous. For example, Riemann discovered the idea of integral (now known as Riemann integral) only after getting motivated by Fourier series. Later Lebesgue, inspired by some problems concerning Fourier series, generalized the concept of Riemann integral to that what is now known as Lebesgue integral. For more details see [1].

After Fourier coined his idea of Fourier series, some related ideas such as Fourier transform, discrete Fourier transform, Fast Fourier transform also came up during the course of time. Among these it is been said that “Fast Fourier Transform” is one of the best algorithm of all times in science and technology. It was proposed by Cooley and Tukey in 1965 [2] and due to this algorithm “Fourier Transform” is known as the "king of all transforms".

Fourier analysis finds its application in many areas of science and technology such as Electrical engineering, Crystallography, X −ray machines, Harmonic signals, Quantum mechanics, Wave motion, Turbulence, Analysis of stationary signals, Real time signal processing

Although, Fourier Analysis find applications in various fields of science and technology, Fourier representation of functions, with the help of sinusoids(Sine and Cosine functions), suffers two major drawbacks: (i) Sinusoids does not have compact support in "time domain"(although they have perfectly compact support in frequency domain).

This makes them non applicable to non-stationary signals (ii) Fourier representation provides spectral content without the time localization. This means that non - stationary signals whose spectral content varies with time can't be analysed with the help of Fourier Analysis".

Once the above weakness in Fourier representation for non - stationary signals was realized by the Applied mathematicians, they started to modify it. In this context the first modification came by Gabor in 1946, who was studying the representation of a communication signal in a time - frequency plane by using oscillatory basis functions. He modified the concept of Fourier transform to "Short Time Fourier Transform" (STFT) by using a Window - Gaussian function. The trick behind the STFT is to represent the signal by using a time - localized window and then doing the analysis for each segment. This modification provides a true time - frequency representation of the signal as in this case we compute Fourier transform for every window (i.e, time localized) segment of the signal . One year later in 1947, Jean Ville proposed a similar transform, Winger - Ville transform for the representation of energy of a signal in the time frequency plane . In fact during the period 1940 to 1970 , many similar transforms were proposed. To cite a few: Cohen Distribution, Wigner- Ville-Rihaczek Distribution etc. All the above Windowed Fourier transforms differ only as far as the choice of the window function is concerned.

Although introduction of Windowed Fourier transforms served some purpose, but when we have to analyse a signal which has high frequency components with short time spans or low frequency components with long time spans, then we need a narrow window to handle first case and a wider window to analyse the second case, and since a Windowed Fourier transform uses only a single window to analyse the entire signal, Windowed Fourier transform is inadequate to handle such signals .Thus, using of a signal window function for the entire signal is a major drawback in Windowed Fourier transform.

In 1970's a geophysical engineer at the French oil company Elf Acquitaine, J. Morlet, was analysing a signal which had low frequency components with long time spans and high frequency components with short time spans. As said above, Windowed Fourier transforms can't handle this situation, so Morlet tried to discover some thing new to handle this situation and he came up with a brilliant idea of using different window functions for analysis different frequency bands in a signal". Also, the different window functions that he used to analyse a signal of above type, are derived from a single function - Gaussian function - by dilation and compression. Due to the " small and oscillatory" nature of these window functions he named them as "wavelets of constant shape." Following the tradition of Mankind of offering opposition to every new idea ", Morlet, just like Fourier, faced much criticism from his contemporaries. Morlet, in his search to find a mathematical rigorous foundation to his ideas, met a theoretical Physicists of quantum mechanics, Grossman, and discussed his ideas with him. Grossman, in Morlet's work, find some thing similar to the coherent states formalism , a technique he had been using in Quantum mechanics, and so he shown a lot of interest in it. After some time, Grossman succeeded in formalising Morlet's ideas and also devised an exact inversion formula for "Morlet's integral transform " and did a lot of applications together with Morlet . In the spring of 1985, Y. Meyer, while waiting on a line to photocopy some papers, heard about Grossman and Morlet's work . After going into their work, Meyer recognised that Morlet and Grossman's analysis and inversion formula is actually a rediscovery of a formula in Harmonic Analysis introduced by A. Calderan in 1960's. Y. Meyer not only recognised the similarity of Morlet and Grossman's work with A .Calderan but also found that there is a great deal of redundancy in wavelets .

Inspired by this, Meyer started working for developing wavelets with better localization properties and he succeeded in producing an "orthogonal wavelet basis" with nice time and frequency localization. Surprisingly, again the Meyer's orthogonal wavelet basis turned out to be a rediscovery of J.O. Stromberg - a

Harmonic analyst - who discovered the same basis five years earlier to Meyer.

It is rather more surprising that the art of constructing orthogonal basis did not start from Stromberg or Meyer but it dates back to 1909, when Alfred Haar, a German Mathematician, constructed an orthonormal wavelet basis although at that time the name wavelet was not in use. Later on It was also discovered that Haar's work of constructing orthonormal basis was expanded by Paul Levy in 1930, when he was engaged in studying "Brownian motion" and also independently by Paley and Littlewood.

Meanwhile Daubechies, a student of Grossman, introduced the concept of "wavelet Frames " for the purpose of discretizing the time and scale parameters of the wavelet transform. This new concept offered more liberty in the matter of "choice of basis", although at the cost of some redundancy .

In 1986, the concept of Multiresolution Analysis for discrete wavelet transform was introduced by Mallat and Meyer. This development, later in 1988 became Ph.D. thesis of Mallat. The idea of Mallat was: 'Decomposition of a discrete signal into its dyadic frequency bands by a series of high pass and low pass filters to compute its DWT from the approximation of these various scales ." It is worth noting that the same ideas were familiar to electrical engineers under the name of " quadrature Mirror Filters " (QMF) and sub band filtering, for about 20 years earlier to Mallat .

In 1988 with the development of Daubechies wavelet orthonormal basis of compactly supported wavelets, the foundations of "Modern wavelet theory" were laid. In last 28 years many new wavelet families such as Daubechies wavlet family, Coiflet wavelet family, Block spline semi - orthogonal wavelet family, Battele - Lemarie's wavelet family, Biorthogonal wavelets of Cohen family, Shannon wavelet family, Meyer's wavelet family, and MRA algorithms have been introduced. For more details on historical development of wavelet analysis , see [3] and the references therein.

Among these families of wavelets, Shannon wavelets offer some more specific advantages, which are usually missing in the others such as: Shannon wavelets are infinitely differentiable, Shannon wavelets are analytically defined, Shannon wavelets are shapely bounded in the frequency domain, Shannon wavelets enjoy a generalization of the Shannon sampling theorem, Shannon wavelets give rise to the connection coefficients which can be analytically defined for any order derivatives, while for the other wavelet families they are computed only numerically that too for the lower order derivatives only.

In the last three decades wavelets find applications in various areas of science and technology. To cite a few: Data compression, Denoising, Source and Channel Coding, Biomedical Engineering, Non destructive Evaluation, Study of Distant Universe, Wavelet Networks, Zero Crossing Representation, Fractals, Turbulence Analysis, Financial Analysis, Medicine, Seismology, Computer graphics, Digital communication, Pattern recognition, Approximation theory, Sampling theory, Statistics, Numerical analysis, Operator theory, Computer vision, Differential equations, Natural scenes, Mammalian visual systems.

As it is well known fact "nothing is perfect in this world", wavelets too face some problems while dealing with objects in more than one dimension. So in last 15 years or so, some new concepts emerged. To name a few: Multi directional wavelets, Complex wavelets, Curvelets, Shearlets, Composite wavelets, Bandelets, Grouplets.

We denote the space of all measureable functions f on $\mathbb{R}$ satisfying $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$ by $\mathbb{L}^1$ or by $\mathbb{L}^1(\mathbb{R})$ and we denote the space of all measureable functions f on $\mathbb{R}$ satisfying $\int_{-\infty}^{\infty}|f(x)|^2\,dx < \infty$ by $\mathbb{L}^2$ or by $\mathbb{L}^2(\mathbb{R})$. For $f \in \mathbb{L}^1$, we call the map which maps f to $\hat{f}$, where $\hat{f}(\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt,\ \omega \in \mathbb{R}$, as the “Fourier transform” and $\hat{f}$ as the Fourier transform of f. If $f \in \mathbb{L}^2$, then Fourier transform $\hat{f}$ of f is defined as $\hat{f}(\omega) = \mathbb{L}^2 - \lim_{N\to\infty}(2\pi)^{-1}\int_{-N}^{N} f(t)exp(-ixt)\,dt$, where the expression like $f(t) = \mathbb{L}^2 - \lim_{n\to\infty} f_n(t)$ means $lim_{n\to\infty}\|f_n - f\|_2 = 0$. Also, we denote $\mathbb{Z}_{\boldsymbol{N}} = \{0,1,2, \dots\dots\dots . N-1\}$, equipped with the $\sigma -$Algebra as the set of all of its subsets and the " measure" as counting measure, so that every function $f : \mathbb{Z}_N \to \mathbb{C}$ is measureable and $\mathbb{L}^1(\mathbb{Z}_N) =$ the set of all functions from $\mathbb{Z}_N$ to $\mathbb{C}$ **.** If $f \in \mathbb{L}^1(\mathbb{Z}_N)$ then " discrete Fourier transform " of f is denoted by Df and is a function from $\mathbb{Z}_N$ to $\mathbb{C}$ defined as $(Df)(n) = \sum_{k=0}^{N-1} f(k)e^{-2\pi ikn/N}$. The map $F: \mathbb{L}^1(\mathbb{Z}_N) \to \mathbb{L}^1(\mathbb{Z}_N)$ defined as $F(f) = Df$ is known as "Discrete Fourier transform operator ".

A sequence of closed subspaces $\left(V_j\right)_{j\in z}$ in $\mathbb{L}^2(\mathbb{R})$ is known as MRA if it satisfies the following properties: (i) $(0) \subset \dots \subset V_{-2} \subset V_{-1} \subset V_{-0} \subset V_1 \subset V_2 \subset \dots \subset \mathbb{L}^2(\mathbb{R})$ (ii)$\bigcap_{j\in\mathbb{Z}} \mathrm{V}_j = \{0\}$ (iii) $\overline{\bigcup_{j\in\mathbb{Z}} \mathrm{V}_j} = \mathbb{L}^2(\mathbb{R})$ (iv) $f(t) \in V_j$ if and only if $f(2t) \in V_{j-1.}$ (iv) There exists $\varphi \in \mathbb{L}^2(\mathbb{R})$, called scaling function, such that $\{\varphi(t-n): n \in \mathbb{Z}\}$ is an orthonormal basis for V_0. The function φ is known as “scaling function of MRA”. If $\{V_j\}_{j\in\mathbb{Z}}$ is an MRA with scaling function φ, it can be shown that $\mathbb{L}^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n$, where $W_n = V_n^{\perp}$ in V_{n+1} This means that every MRA produces an orthogonal direct sum decomposition of the Hilbert space $\mathbb{L}^2(\mathbb{R})$. For $f \in \mathbb{L}^2(\mathbb{R});\ j, k \in \mathbb{Z}$, we denote $f_{j,k}(t) = 2^{\frac{j}{2}} f\left(2^j\ t - k\right)$. If $\{V_j\}_{j\in\mathbb{Z}}$is an MRA and $\psi \in V_1$, then we say ψ is a mother wavelet if $\{\psi(t-n): n \in \mathbb{Z}\}$ is an orthonormal basis of $W_O = V_0^{\perp}(\mathrm{in}V_1)$. If ψ is a mother wavelet, then we call $\psi_{j,k}$ *'s* as wavelets and the collection $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ as "orthonormal wavelet basis with mother wavelet ψ ". If $\left(\{V_j\}_{j\in\mathbb{Z}},\varphi\right)$ is an MRA and ψ is a mother wavelet, then it can be shown that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and that the collection $\{\psi_{j,k}: j,k \geq and\ j \geq 0\} \cup \{\varphi_{0,k}: k \in \mathbb{Z}\}$ are both orthonormal basis of $\mathbb{L}^2(\mathbb{R})$. The method of getting a mother wavelet from an MRA $(\{V_j\},\varphi)$ is given by Mother wavelet theorem[1].

Let $h \in \mathbb{L}^2(\mathbb{R})$ be fixed. Then the continuous wavelet transform of $f \in L^2(\mathbb{R})$ induced by h is denoted by $\mathcal{W}_h f$ and is a function from $\mathbb{R}^* \times \mathbb{R}$ to $\mathbb{C}$ defined as $(\mathcal{W}_h f)(a,b) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{+\infty} f(t).\overline{h\left(\frac{t-b}{a}\right)}\,dt$, where $\mathbb{R}^* = \mathbb{R}-\{0\}$. Here, h is usually known as wavelet and $(\mathcal{W}_h f)(a,b)$ are known as continuous wavelet coefficients of f. When the wavelet $'h'$ satisfies the admissibility condition $C_h = \int_{-\infty}^{+\infty}|\omega|^{-\mathbf{1}}\left|\hat{h}(\omega)\right|^{\mathbf{2}} d\omega < \infty$, then it can be shown that if h , $\hat{h}$ are both windows that is, $h,\hat{h} \in \mathbb{L}^2(\mathbb{R})$ and $th, t\hat{h} \in \mathbb{L}^2(\mathbb{R})$), then $\int_{-\infty}^{+\infty} h(t)dt = 0$ and the existence of inversion formula and Parseval's formula for a wavelet transform can be established [1]. Since inversion formula and the property $\int_{-\infty}^{+\infty} h(t)dt = 0$ are very important tools in wavelet applications, so usually the admissibility condition is included in the definition of a wavelet and wavelet transform. If we take $a = a_0^j\,;\, b = kb_0a_0^j$, where $a_0 > 0\,;\, b_0 \in \mathbb{R}\,; j,k \in \mathbb{Z}$, then $(\mathcal{W}_h f)\left(a_0^j, kb_0a_0^j\right)$ is usually denoted by $(\mathcal{W}_h f)(j,k)$ are known as " Discrete wavelet coefficients of f" . We can recover f by Discrete wavelet coefficients of f in a numerically stable manner but if $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$ forms a frame in $\mathbb{L}^2(\mathbb{R})$ [1]. If $\left(\{V_j\}_{j\in\mathbb{Z}},\varphi\right)$ is an MRA with ψ as Mother wavelet, then we know that $\{\psi_{j,k}\}$ is an orthonormal basis of $\mathbb{L}^2(\mathbb{R})$ so that for any $f \in \mathbb{L}^2(\mathbb{R})$, we have $f = \sum_{j,k\in\mathbb{Z}}\langle f,\psi_{j,k}\rangle\psi_{j,k}$. Thus we can know f , by knowing $\langle f,\psi_{j,k}\rangle$ .But usually it takes a lot of time and effort. In 1989 Mallat discovered a fast way to compute these coefficients. The method that he proposed is known as " Fast wavelet transform" algorithm. For a detailed account of this concept see [4].

II. Shannon Scaling Function And Shannon Wavelet

The function $\varphi: \mathbb{R}^* \to \mathbb{R}$ defined as $\varphi(x) = (\pi x)^{-1} sin(\pi x)$, where $\mathbb{R}^* = \mathbb{R}-\{0\}$, known as the Shannon scaling function. It can be seen that $\{\varphi(\cdot - n) : n \in \mathbb{Z}\}$ is an orthonormal family in $L^2(\mathbb{R})$ [1]. Further if we define $V_0 = \{f \in L^2(\mathbb{R}): \hat{f}(\omega) = 0\ for|\omega| > \pi\}$ -the space of band-Limited functions- then by Sampling theorem it can be shown [1] that $\{\varphi(\cdot - n) : n \in \mathbb{Z}\}$ is an orthonormal basis for V_0. For any real function f , and $n,k \in \mathbb{Z}$, if we denote $f_{k,n}(x) = 2^{-2^{-1}n} f(2^n x - k)$. and $V_n = span\{\varphi_{k,n} : k \in \mathbb{Z}\}$, then it can be seen that $V_n = \{f \in L^2(\mathbb{R}): \hat{f}(\omega) = 0\,, |\omega| > 2^n\pi\}$ and $\{V_n: n \in \mathbb{Z}\}$ is Multiresolution Analysis (MRA), thus, the name Scaling function to φ is justified.. The MRA obtained above is known as Shannon MRA . Mother wavelet of Shannon MRA is known as " Shannon wavelet".

The Shannon scaling function φ in the Fourier domain is given by $\hat{\varphi}(\omega) = \frac{1}{2\pi}\chi(\omega + 3\pi)$ and the Shannon wavelet ψ in the Fourier domain is given by $\hat{\psi}(\omega) = \frac{1}{2\pi}e^{-i\omega}[\chi(2\omega) + \chi(-2\omega)]$, where χ is the characteristic function of $[2\pi, 4\pi]$. Also, Shannon wavelet ψ in the real domain is given by $\psi(x) = \left[sin\left(\pi(x - 2^{-1})\right) - sin\left(2\pi(x - 2^{-1})\right)\right][\pi(x - 2^{-1})]^{-1}$. For $n,k \in \mathbb{Z}$ and Shanon wavelet ψ , the functions $\psi_{n,k}$ are known as Shannon wavelets and for the Shannon scaling function φ, the function $\varphi_{n,k}$ are known as Shannon scaling functions. Explicit expressions for $\psi_{n,k}$ and $\varphi_{n,k}$ and their Fourier transforms are given as: $\varphi_{k,n}(x) = 2^{2^{-1}n} sin\left(\pi(2^n x - k)\right)[\pi(2^n x - k)]^{-1}$, $\hat{\varphi}_{k,n}(\omega) = 2^{-n2^{-1}}(2\pi)^{-1}e^{2^{-n}-i\omega k}\chi(2^{-n}\omega + 3\pi)$, $\psi_{k,n}(x) = [\pi(2^n x - k - 2^{-1})]^{-1} \times 2^{2^{-1}n}\left[sin\left(\pi(2^n x - k - 2^{-1})\right) - sin\left(2\pi(2^n x - k - 2^{-1})\right)\right]$, and $\hat{\psi}_{k,n}(\omega) = -(2\pi)^{-1}2^{-n2^{-1}}e^{-i2^{-n}\omega(k+2^{-1})}[\chi(\omega 2^{n-1}) + \chi(-\omega 2^{n-1})]$ [5]. Fractional derivatives of Shannon wavelets and Shannon scaling functions have been discussed by Cattani in [6].

III. Properties of Shannon scaling functions and Shannon wavelets

In this section, we record some properties of Shannon wavelets, $\psi_{k,n}$ and Shannon scaling functions , $\varphi_{k,n}$ form [5]. Let $m,n,k,h \in \mathbb{Z}$. Then (i) $\langle\psi_{k,n},\psi_{n,m}\rangle = \delta^{nm}\delta_{nk}$, where δ^{nm} and δ_{nk} are Kroenecker symbols (ii) If $0 \leq m,k,h$ we have $\langle\varphi_{k,0},\psi_{h,m}\rangle = 0$ but if $m < 0$, then $\langle\varphi_{k,0},\psi_{h,m}\rangle = 0$ does not vanish in general. (iii) $\langle\varphi_{k,0},\psi_{h,0}\rangle = \delta_{kh}$ (iv) $\psi_{k,n} = 0$ for $x = 2^{-n}(k + 2^{-1} \pm 3^{-1})$ and $n \in \mathbb{N}$. (v) $\lim_{x\to 2^{-n}(h+2^{-1})}\psi_{k,n}(x) = -2^{n2^{-1}}\delta_{h,k}$ (vi) $\lim_{x\to\pm\infty}\varphi_{k,n}(x) = 0$ (vi) $\lim_{x\to\pm\infty}\psi_{k,n}(x) = 0$

(vii) For a fixed x_0 ,

(a) $\varphi_{k+1,n}(x_0) < \varphi_{k,n}(x_0)$

(b) $\varphi_{k+1,n}(x_0)\left(\varphi_{k,n}(x_0)\right)^{-1} = (2^n x - k)(2^n x - k + 1)^{-1}$

(c)$\psi_{k+1,n}(x_0)\left(\psi_{k,n}(x_0)\right)^{-1} =$
$(2^{n+1}x-2k-1)(2\,sin[\pi(2^n x-k)]-1)\times\ (2^{n+1}x-2k-3)^{-1}\times(2\,sin[\pi(2^n x-k)]+1)^{-1}$
(d) $\psi_{k,n+1}(x_0)\left(\psi_{k,n}(x_0)\right)^{-1}=\sqrt{2}(2^{n+1}x-2k-1)\times(\cos[\pi(2^{n+1}x-k)]-\sin[2\pi(2^{n+1}x-k)])\times$
$(2^{n+2}x-2k-1)^{-1}\ (\cos[\pi(2^n x-k)]-\sin[2\pi(2^n x-k)])^{-1}$
(vii) $\lim_{x\to\infty}\psi_{k+1,n}(x)\left(\psi_{k,n}(x)\right)^{-1}<1$
(viii) $\lim_{x\to 2^{-n}(k+2^{-1})}\psi_{k+1,n+1}(x)\left(\psi_{k,n}(x)\right)^{-1}=2\sqrt{2}[cos(k\pi)-sin(2k\pi)]\left((2k-1)\pi\right)^{-1}$
(ix) Maximum value of $\varphi_{k,0}$ is 1 at k (x) Maximum value of $\psi_{k,n}$ is $2^{\frac{n}{2}}\frac{3\sqrt{3}}{\pi}$ at $2^{-n}(k+6^{-1})$ or $3^{-1}2^{-n-1}(18\,k+7)$ (xi) Minimum value of $\varphi_{k,0}$ is approximately $sin\sqrt{2}\pi\left(\sqrt{2}\pi\right)^{-1}$ at $x=k-1\pm\sqrt{2}$
(xii) Minimum value of $\psi_{k,n}$ is $-2^{n2^{-1}}$ at $2^{-n}(2k+1)$.

IV. Reconstruction of Functions By Shannon Scaling Functions And Shannon Wavelets

In this section, we record two results from [7] showing the reconstruction of a function with the help of Shannon scaling functions and Shannon wavelets .The first result is the famous Shannon sampling theorem and the second is generalisation of this theorem.

4.1 Theorem [7]: (Shannon Sampling theorem): If $f\in\mathbb{L}^2(\mathbb{R})$ and $supp(\hat{f})\subset[-\pi,\pi]$, then the series $\sum_{k=-\infty}^{\infty}\alpha_k\varphi_{k,0}$ converges uniformly to $f, on\ \mathbb{R}$, and $\alpha_k=f(k)$.
(Shannon Generalization) If $f\in B_\psi$ and $supp\ \hat{f}\subseteq\mathbb{R}$, then the series $\sum_{h=-\infty}^{\infty}\alpha_h\varphi_{h,0}+\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\beta_{k,n}\psi_{k,n}$ converges to f. In particular, if $supp\hat{f}\subset[2^{-N}\pi,2^N\pi]$, then $\sum_{h=-\infty}^{\infty}\alpha_h\varphi_{h,0}+\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\beta_{k,n}\psi_{k,n}$ converges to f, where ψ is Shannon wavelet and B_ψ is the space of all those functions f such that $\alpha_k=\int_{-\infty}^{+\infty}f(x)\varphi_{k,0}(x)dx=\int_{-\pi}^{+\pi}\hat{f}(\omega)e^{i\omega k}$ is finite and $\beta_{k,n}=\int_{-\infty}^{+\infty}f(x)\psi_{k,n}(x)dx=-2^{-n2^{-1}}$ $\int_{2^n\pi}^{2^{n+1}\pi}\hat{f}(\omega)e^{i\omega(k+2^{-1})2^{-n}}d\omega-2^{-n2^{-1}}\times\ \int_{-2^{n+1}\pi}^{-2^n\pi}\hat{f}(\omega)e^{i\omega(k+2^{-1})2^{-n}}d\omega$ is finite.

If we fix an upper bound in the series involved in the Shannon generalisation in such a way that we have approximation of f as

$$f(x)\cong\sum_{h=-k}^{k}\alpha_h\varphi_{h,0}(x)+\sum_{n=0}^{N}\sum_{k=-s}^{s}\beta_{k,n}\psi_{k,n}\,(x) \tag{1}$$

then the approximation error according to [8] satisfies:
$\left|f(x)-\sum_{h=-k}^{k}\alpha_k\varphi_{h,0}(x)+\sum_{n=0}^{N}\sum_{k=-s}^{s}\beta_{k,n}\psi_{k,n}\,(x)\right|\leq\left|f(-k-1)+f(k+1)-3\sqrt{3}\pi^{-1}\left[f\left(2^{-N-1}(-s-\right.\right.\right.$ *2−1+ f2−N−1s+1.5.* In [7], the absolute values of approximation errors for some functions for given values of k and N have been calculated. For example for the function $f(x)=e^{-4x^2}cos(2\pi x)\ x\in\mathbb{R}$, the absolute value of the approximation error is computed for $N=3$ and $|k|\leq 3$ and is found to be less than 7%.

V. Reconstruction of the derivatives by using Shannon scaling functions and Shannon wavelets

Let $f\in L^2(\mathbb{R})\cap C^P(\mathbb{R})$ for sufficiently high value of p.Then by Shannon generalisation we can write $\frac{d^l}{dx^l}f(x)=\sum_{h=-\infty}^{\infty}\alpha_h\frac{d^l}{dx^l}\varphi_{h,0}(x)+\quad\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\beta_{k,n}\frac{d^l}{dx^l}\psi_{k,n}(x)$. This equation suggests that we may know $\frac{d^l}{dx^l}f(x)$ provided we know $\frac{d^l}{dx^l}\varphi_{h,0}(x)$ and $\frac{d^l}{dx^l}\psi_{k,n}$. These two quantities can be known easily for $l=1,2$, [7], but for higher values of l the direct computation of the above two quantities is very difficult. Cattani in [7] provided following expressions for calculating these quantities:

5.1 Theorem [7]: Under Usual notations, we have
(i) $\frac{d^l}{dx^l}\varphi_{h,0}(x)\ =\ \sum_{k=-\infty}^{\infty}\lambda_{hk}^{(l)}\varphi_{k,0}^{(x)}$
(ii) $\frac{d^l}{dx^l}\psi_{h,m}(x)=\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty,},\gamma_{hk}^{(l)mn}\psi_{k,n}(x)$
(iii) $\langle\frac{d^l}{dx^l}\varphi_{h,0}\,,\frac{d^p}{dx^p}\psi_{h,m}\rangle=\ 0$
Here in this theorem the quantities $\lambda_{kh\text{h}}^{(l)}$, $\gamma_{hh\text{kh}}^{(l)nm}$ are known as Connection coefficients or Cattani connection coefficients and are given by

(i) $\lambda_{kh\mathrm{h}}^{(l)} = \langle \frac{d^l}{dx^l}\varphi_{k,0}, \varphi_{h,0}\rangle = (-1)^{k-h}\frac{i^l}{2\pi}\sum_{s=1}^{l}\frac{l!\pi^s[(-1^s-1)]}{s![i(k-h)]^{l-s+1}} \quad if\ k \neq h$ and

$\lambda_{kh\mathrm{h}}^{(l)} = \langle \frac{d^l}{dx^l}\varphi_{k,0}, \varphi_{h,0}\rangle = \frac{i^l\pi^{l+1}}{2\pi(l+1)}[1+(-1)^l] \quad if\ k = h$

(ii) $\gamma_{k\mathrm{h}}^{(l)nm} = \langle \frac{d^l}{dx^l}\psi_{k,n}, \psi_{k,m}\rangle = \delta^{nm}\{i^l(1-|\mu(h-k)|)\}\frac{\pi^l 2^{nl-1}}{l+1} + (2^{l+1}-1)(1+(-1)^l)\mu(h-k)$

$$\times \sum_{s=1}^{l+1}\left\{(-1)^{[1+\mu(h-k)](2l-s+1)/2}\frac{l!\,i^{l-s}\pi^{l-s}}{(l-s+1)!\,|h-k|^s}(-1)^{-s-2(h+k)}2^{nl-s-1}\right\}\times$$

$\{2^{l+1}[(-1)^{4h+s}+(-1)^{4h+l}] - 2^s[(-1)^{3k+h+l}+(-1)^{3h+k+s}]\} \quad for\ l \geq 1$

and $\gamma_{hhkh}^{(0)nm} = \delta_{kh}\delta^{nm}$, where δ^{nm} is Kroenecker symbol.

Although infinite sums in above expressions suggest that these expressions for the derivatives are not good enough, but practically this is not the case as we can obtain a very good approximation of the derivatives just by considering only a first few terms of the series, because these series involve Shannon wavelets which are mainly localized in a short range interval. Also, if we consider only a few terms in above equation**s**, we have error estimates as stated in [6] which read as:

(i) $\left|\frac{d^l}{dx^l}\varphi_{h,0}(x) - \sum_{k=-N}^{N}\lambda_{hhk}^{(l)}\varphi_{k,0}(x)\right| \leq \left|\lambda_{h(-N-1)}^{(l)} + \lambda_{h(N+1)}^{(l)}\right|$

(ii) $\left|\frac{d^l}{dx^l}\psi_{h,m}(x) - \sum_{n=0}^{N}\sum_{k=-S}^{S}\gamma_{kh}^{(l)nm}\psi_{k,n}(x)\right| \leq \left|2^{l(m-1)+2^{-1}m}\frac{3\sqrt{3}}{\pi}\left[\gamma_{h(-s-1)}^{(l)11} + \gamma_{h(s+1)}^{(l)11}\right]\right|$. In [6] Cattani has provided some recursive formulae for connection coefficients:

(i) $\lambda_{k\mathrm{h}}^{(l+1)} = \frac{l+1}{k-h}\lambda_{kh}^{(l)} - (-1)^{k-h}\frac{i^l\pi^{l+1}}{k-h}[-1^l+1]\ if\ k \neq h$

(ii) $\lambda_{kh}^{(l+1)} = i\pi\frac{l+1}{l+2}\lambda_{kh}^{(l)} + \frac{-(i)^{l+1}\pi^{l+1}}{l+2}\ if\ k = h$

(iii) $\gamma_{hhkh}^{(l)nn} = 2^{l(n-1)}\gamma_{kh}^{(l)11}$

In [9] some identities satisfied by connection coefficients have been proved.

VI. Applications of Shannon Wavelet Analysis to Integro- Differential Equations

Since many real world situations are modelled by integro-differential equations- it becomes necessary to formulate suitable methods to solve such equations, and since it is not always possible to get analytical solutions of these equations ,applied mathematicians have been busy to develop methods of getting efficient numerical solutions of these equations over the years. In this direction, wavelet analysis is playing a huge role and, in particular, Shannon wavelet. Consider the integro - differential equation $A\frac{du}{dx} = B\int_c^d k(x,y)u(y)dy + Cu(x) + q(x)$, where $A, B \in \mathbb{R}$; $k(x,y), q(x)$ are given and u is unknown function.

Cattani in [8] proposed a method of solving this equation numerically by using Shannon wavelet analysis with Petrov Galerkin method in the case $k(x,y) = f(x) \times g(x)$ with $f, g \in \mathbb{L}^2(\mathbb{R})$, $c = -\infty$, $d = \infty$ and $A = B = C = 1$. In this paper the author also illustrated the method by discussing some specific examples.

Maleknejad and Attary in [10] proposed a method of obtaining the numerical solution of the equation by using Shannon wavelet analysis with collocation approach when $c = a \in \mathbb{R}$; $d = b \in \mathbb{R}$,$A = B = C = 1$ and $q \in l^2(\mathbb{R})$. In this paper, the author also discussed error analysis and illustrated the method by supplying some examples and compared their results with the methods of solving them as proposed in [11] or [12]. For the example $f'(t) - f(t) - \int_0^1 e^{st}f(s)ds = \frac{1-e^{t+1}}{t+1}; f(0) = 1$, the authors have shown that Shannon approximation produced better numerical results in comparison to Hybrid Legendre Block Plus functions approximation of the problem as proposed in [11] and for the example $f'(t) - f(t) - \frac{1}{(ln\ 2)^2}\int_0^1 \frac{t}{s+1}f(s)ds = \frac{1}{t+1} - \frac{t}{2} - ln(t+1)$; $f(0) = 0$, the authors calculated maximal error and compared the method with the method Proposed in [12], where the problem have been solved by a method based on Whittaker Cardinal expansion approach and found that both methods produce nearly equivalent approximation solution for small values of N and M. It has been claimed by Maleknejad and Attary [10] that extremely good numerical results can be achieved for $N \geq 2$ and $M \geq 3$.

VII. Applications of Shannon Wavelet Analysis to Integral Equations

To solve integral equations numerically, traditional quadrature formula methods and spline approximation methods are used by the Applied Mathematicians. While using these methods, it is required to solve systems of linear equations for which usually matrix methods are applied, and if the number of linear equations is too large, we get big matrices and hence we require too many arithmetic operations and huge storage capacity. But if we can replace fully populated transpose matrix by a sparse matrix, we can reduce both number of arithmetic operations and storage capacity. But if we can replace the fully populated matrix by a sparse matrix, we can reduce the both number of operations and the storage capacity. One way of replacing the given matrices by sparse matrices is by using wavelet basis, because the wavelet basis functions are orthogonal to each other. Probabaly the first approach to solve integral equations by wavelet methods is work done by Beylkin et al [13], in 1991. Various contributions were made by many Applied mathematicians in which Daubechies wavelets, adaptive Battle-Lemarie wavelets, Hermatic type trigonometric wavelets; Haar wavelet, linear B-Splines, Walsh functions, Chen and Albert wavelets were used. For example one can see [14], [15] [16], [17],[18] ,[19] ,[20] ,[21], [22], [23], [24], [25], [26].

But the Shannon wavelets have been used by Danesfahani et al [27] to solve Fredholm integral equations of first and second kind. In this paper the authors used Shannon wavelet analysis with the method of moments to present methods of solving Fredholm integral equations of first and second kind. The method that they proposed for solving Fredholm integral equation of first kind $-\int_a^b k(s,t)x(t)\,dt = y(s)$, where $k(s,t) \in L^2([a,b) \times [a,b)]$; $y(s) \in L^2([a,b))$ are known functions and $x(t)$ is unknown$-$ is:

Step 1: Assume $x(t) \in V_j$ for some $j \in \mathbb{Z}$ so that one can write $x = \sum_{k\in\mathbb{Z}} c_k\, \varphi_{k,j}$

Step 2: Assume $n = 2^j$ and approximate x as $x = \sum_{k=0}^{n} c_k\, \varphi_{k,j}$ ...(a), so that we have

$$\sum_{k=0}^{n} c_k \int_a^b k(s,t)\, \varphi_{k,j}\,(t)\, dt = y(s).$$

Step 3: Choose $s_1\,, s_2, \ldots\,, s_n$ in $[a,b)$ so that on substituting for s equation (a) of step 2, we get a system of n equations: $\sum_{k=1}^{n} c_k \int_a^b k(s_i,t)\, \varphi_{k,j}\,(t)\, dt = y(s_i)\,; (1 \le i \le n).$

Step 4: Solve this system of n linear equations in n unknown $c_1\,, c_2, \ldots\,, c_n$ and put in equation (a) of step 2, thus getting approximate solution.

Similar method for solving Fredholm integral equations of second kind can be seen in the same paper [27]. In the same paper [27], the above methods have been illustrated by applying to some examples and it has been found that the proposed methods have a high accuracy and efficiency in the case in which the kernel function is of the form $k(s,t) = H_0^2(\alpha|s-t|)$, where H_0^2 is the Hankel function of 2nd kind of order zero.

VIII. Applications of Shannon Wavelet Analysis To Differential Equations

It is well known that most of the real world problems are usually get modelled in the form of differential equations and exact solution of differential equations are either hard or sometimes impossible to find. So, Applied Mathematicians are always remain busy for proposing new and efficient numerical methods for solving differential equations. For example one can see [28], [29], [30], [31], [32], [33], [17]. In [9] Shi and Li used 1-periodized Shannon wavelets for solving higher order differential equations. The authors of the paper defined 1- periodic Shannon wavelets and 1- periodic Shannon scaling functions as: If φ, ψ be the Shannon scaling function and Shannon wavelet, then for any $n, k \in \mathbb{Z}$, the 1- periodic Shannon scaling functions, $\tilde{\varphi}_{k,n}$ and 1- periodic Shannon wavelets, $\tilde{\psi}_{k,n}$ are defined as $\tilde{\varphi}_{k,n}(x) = \sum_{r\in\mathbb{Z}} \varphi_{k,n}(x+r)\,, x \in \mathbb{R}$ and $\tilde{\psi}_{k,n}(x) =$
$\sum_{r\in\mathbb{Z}} \psi_{k,n}(x+r)\,, x \in \mathbb{R}$. In the same paper, the authors have proved many properties of 1- periodic Shannon wavelets and 1- periodic Shannon scaling functions similar to the ones we have recorded for Shannon wavelets and Shannon scling function in searlier sections. As with the Shannon scaling function and Shannon wavelet, we have connection coefficients $\tilde{\gamma}_{k,n;l,m}^{(s)}$ for 1-periodised Shannon scaling function and Shannon wavelet too, which are defined as $\tilde{\gamma}_{k,n;l,m}^{(s)} = \langle \frac{d^s}{dx^s}\tilde{\psi}_{k,n}, \tilde{\psi}_{l,m}\rangle$. These connection coefficients are connected to the Cattani connection coefficients by the relations: (i) $\tilde{\gamma}_{k,n;l,m}^{(s)} = 0 \; if \; n \ne m$ (ii) $\tilde{\gamma}_{k,n;l,n}^{(s)} = \sum_{r\in\mathbb{Z}} \gamma_{k,n;l-2^n r,n}^{(s)}$ (Here $\gamma_{k,n;l,m} = \gamma_{kl}^{nm}$). By using connection coefficients, we have: For any $u \in \mathbb{L}^2[0,1]$ which is sufficiently differentiable, $\frac{d^s}{dx^s}u(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \tilde{\beta}_{k,n} \left[\sum_{l=0}^{2^n-1} \left(\sum_{r\in\mathbb{Z}} \tilde{\gamma}_{k,n;l-2^n r,n}{}^{(s)}\right) \tilde{\psi}_{l,n}\;\;(x)\right]$ [9].

To solve higher order differential equations the authors of [9] used the approximations of u and $u^{(s)}$ as

$$u(x) = \alpha_0 + \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \tilde{\beta}_{k,n}\, \tilde{\psi}_{l,n}\;\;(x)\,;\; x \in [0,1]$$

and $$\frac{d^s}{dx^s}u(x) = \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \tilde{\beta}_{k,n} \left[\sum_{l=0}^{2^n-1} \left(\sum_{r=-M}^{M} \tilde{\gamma}_{k,n;l-2^n r,n}{}^{(s)}\right) \tilde{\psi}_{l,n}\;\;(x)\right] ; x \in [0,1]$$ and

demonstrated the method by solving BVP: $y^{(12)}(x) - y(x) = -12(2x cos x + 11 sin x); -1 \le x \le 1$ subject to the boundary conditions: $y(-1) = y(1) = 0$; $y'(-1) = y'(1) = 2sin(1)$;$y''(-1) = -y''(1) = -4co\,s(1) - 2\,sin(1)$;$y'''(-1) = y''{}'(1) = 6cos 1 - 6\,sin 1$; $y^{(4)}(-1) = -y^{(4)}(1) = 8\,cos\,1 + 12\,sin(1)$;$y^5(-1) = y^{(5)}(1) = -20\,cos(1) + 10\,sin(1)$. They also estimated the error by the formula $Error = [(\sum_{k=1}^{X}(y_e(k) - y(k)^2))x^{-1}]^{2^{-1}}$ where X = No.of collocation points; y is the approximation solution and y_e is the exact solution and got error estimates for $N = 4,5,6,7$ and $M = 50$. The error estimates show that the method is very efficient. Shi and Li also demonstrated the method by taking another BVP and compared its results to those of [32]. Although the method demonstrated in [9] is by taking problems involving ordinary differential equations, but the authors strongly believe that the method is also applicable to partial differential equations.

IX. Method of solving inverse heat conduction problems by using Shannon wavelet analysis

In many industrial problems, surface of a body is inaccessible, but we need to measure its (surface) temperature. In these problems, we measure the temperature of the body by using temperature history at a fixed point inside the body. This process is known as " Inverse Heat Conduction Problem (IHCP)" and is a subject of great interest in recent years. The standard Inverse Heat conduction Problem is : $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$; $x>0, t>0$; $u(x,0)=0$; $x\geq 0$; $u(1,t)=g(t)$; $t\geq 0$; $u(x,t)$remains bounded as $x\to\infty$. The solution of the problem IHCP has been a subject of great interest and has been discussed by many Applied Mathematicians. For example, Reginska and Elden [34] solved the problem by using a wavelet - Galerkin method; Wang in al [35] presented a multiresolution method for solving the problem; Scidman and Elden [36] proposed an optimal filtering method for the problem; Carano [37] and Qion et al. [38] also presented methods of solving the problem; Carasso [39] and Fu [40] used Tikhonov methods to analyse IHCP; Mollification method has been used by Musio [41] and by Hao et al [42] to analyse IHCP; Wavelet Galerkin methods have been used by Elden et al in [43],[44],[45],[46] and [47] to discuss IHCP; Cheng et al in [48] used a modified Ti Khonov method for the IHCP and Tautenhahn [49] obtained optimal approximations for IHCP. The above cited methods mostly dealt with IHCP in semi unbounded domain, however, some methods have also been applied to IHCP with bounded domain . For example, Chen et al [50] studied IHCP in a rectangular plate by applying hybrid numerical algorithim of Laplace Transform technique; Busby and Trujillo in [51] discussed IHCP in a slab by using dynamical programming method; Alifanov and kerov [52] and Louahlia - Gualou et al [53]discussed IHCP in a cylinder. Also the non standard Inverse heat conduction problem: $\frac{\partial u}{\partial t}+\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$; $x>0\; and\; t>0$; $u(x,0)=0$; $x\geq 0$; $u(1,t)=g(t)$; $t\geq 0$; $u(x,t)$ remains bounded as $x\to\infty$; $t\geq 0$; which is connected to IHCP appears in many applied subjects and is a subject of study by many applied mathematicians. For example, Xiong et.al [54],[55] investigated the problem by central difference method ; Reginska [56] presented solution of the problem in the interval [0,1) by using wavelet dual least square method, in which they used Meyer wavelet.

The problem has also has been solved by using wavelet dual least square method generated by the Shannon wavelets [57]. In this method, firstly the functions $u(x,.), g(.), u(o,.)=f(.)$ are extended from the domain [0,∞) to ℝ by putting $u(x,t)=0$, $g(t)=0$ and $u(0,t)=0\; for\; t<0$, so that we can work in $\mathbb{L}^2(\mathbb{R})$. After that it was assumed that for given 'g', the solution 'u' of the problem exists and satisfies an a - priori bound: $\|u(0,.)\|_p\leq E$, $p\geq 0$, where $\|(.)\|_p$ stands for $\left(\int_{-\infty}^{\infty}(1+\xi^2)^p\left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}=\|f\|_p$. Further since g is to be measured by the thermocouple, it was assumed that there is some function $g_\delta\in\mathbb{L}^2(\mathbb{R})$ satisfying $\|g_\delta(.)-g(.)\|_{L^2(\mathbb{R})}\leq\delta$, where δ represents a bound on the measurement error. Further it was also assumed that $\|u(x,.)\|$ is bounded which will assure the uniqueness of the solution [58]. Finally after proving various results the authors of [57] presented the method of solving the problem as: Choose two families of subspaces$\{V_j\}$ and $\{Y_j\}$ of $L^2(\mathbb{R})$ such that $V_j\subset R(K^*)$ and $K^*Y_j = V_j$. Then choose y_λ satisfying $K^*y_\lambda = k_\lambda\ \Psi_\lambda$, $\|y_\lambda\|=1$, where $\{\Psi_\lambda\}_{\lambda\in\tilde{I}_j}$ is an orthogonal basis of V_j, so that we have following approximation $u_j=\sum_{\lambda\in\tilde{I}_j}\langle g, y_\lambda\rangle(k_\lambda)^{-1}$. For noisy data g_δ, we take wavelet dual least squares approximation solution of the problem in the interval $0\leq x<1$ as $P_J u^\delta(x,t)=u_J^\delta=\sum_{\lambda\in\tilde{I}_J}\langle g_\delta, y_\lambda\rangle\frac{1}{k_\lambda}\Psi_\lambda=\sum_{\lambda\in\tilde{I}_J}\langle u^\delta,\Psi_\lambda\rangle\Psi_\lambda$.

Here in this method, for $x\in[0,1)$, K_x is the operator on $\mathbb{L}^2(\mathbb{R})$ defined as $K_x\, u(x,.)=g(.)$ and K_x^* is the adjoint of K_x. Also, $P_J: L^2(\mathbb{R})\to V_J$ is defined as $P_J s=\sum_{\lambda\in I_J}\langle s,\Psi_\lambda\rangle\Psi_\lambda$, $\forall\, s\in L^2(\mathbb{R})$. The properties of these operators have been recorded in [57].

Finally, following result is presented by the authors of [57] which measures the error:

9.1 Theorem[57] Let u be the exact solution of the problem and $P_J u^\delta$ is given as in the above proposed method. Let $g_\delta(t)$, satisfy the previously imposed assumption at $x=1$. Let $J=\log_2[2\pi^{-1}(ln(\delta^{-1}E(ln\,E-ln\delta)^{-2p}))^2]$. Then for any fixed $x\in[0,1)$, $\left\|u(x,.)-P_J u^\delta(x,.)\right\|\leq E^{1-x}\delta^x\left(\ln\frac{E}{\delta}\right)^{-2p(1-x)}\left(\sqrt{e}+1+o(1)\right)\; as\; \delta\to 0.$

Note that if $p=0$ and $0<x<1$, then the conclusion in the above theorem is Holder stability estimate given by $\left\|u(x,.)-P_j u^\delta(x,.)\right\|\leq(\sqrt{e}+1)E^{1-x}\delta^x$; if $p>0$ and $0\leq x<1$, then the conclusion in the above theorem is logarithmic Holder stability estimate and **i**f $p>0$ and $x=0$, then the conclusion in the above theorem becomes $\|u(0,.) - P_j u^\delta(0,.)\|\|u^\delta(x,.)\| \le E\left(\ln\frac{E}{\delta}\right)^{-2p}\left(\sqrt{e}+1+o(1)\right)\ as\ \delta \longrightarrow 0$, which is similar to the convergence estimate in [59]. In general, the a - priori bound E is unknown in practice. In this case, with$J = \log_2[2\pi^{-1}(ln(\delta^{-1}E(ln\,E - ln\delta)^{-2p}))^2]$, we have $\|u(x,.) - P_j u^\delta(x,.)\| \le \delta^x \left(\ln\frac{1}{\delta}\right)^{-2p(1-x)}\left(\sqrt{e}E+1+o(1)\right)\ as\ \delta \to 0$, where E is only a bounded positive constant and it is not exactly necessary known.

It is claimed by Cheng et al [57] that most results putting forward error estimates in the literature are of Holder's type: $\|u(x,.) - v(x,.)\| \le 2E^{1-x}\delta^x$, where E is an a Priori bound of the function $u(0,.)$. Therefore, as $x \to 0^+$, the accuracy of the regularized solution becomes lower .Infact at $x = 0$, we have $\|u(x,.) - v(x,.)\| \le 2E$, i.e at $x = 0$, it tells us only that error is bounded by $2E$ and does not prove convergence. In the discussion made in above lines, by taking suitable value of J, we not only obtain the Holder continuity with $p = 0$ for $0 < x < 1$, but also get a logarithimic Holder convergence estimate with $p > 0$ and $0 < x < 1$. Especially we gained the logarithmic type convergence estimate on the boundary $x = 0$. This is in fact an improvement of various results in [43]

X. Second order approximation of C^2 functions by using Shannon wavelet analysis

The 2nd order approximation of a function $f(x) \in C^2$ in x_0 , is defined as $f(x) \cong f(x_0) + ap(x) + bq(x)$, where $p(x)\,, q(x)$ are chosen in a such a way that $p(x_0) = 0$, $q(x_0) = 0$, $p^{'}(x_0) \neq 0$, $p^{''}(x_0) = 0$, $q^{'}(x_0) = 0$, $q^{''}(x_0) = 0$. In [60], Cattani has established following approximation of C^2 functions by using Shannon wavelet analysis.

10.1 Theorem [60]: Let f be a given function such that for a fixed n, k, in one of the two points $x_\pm = 2^{-n}(k + 2^{-1} \pm 3^{-1})$, it is at least C^2 and $f^{'}(x_+) > 0$, $f^{'}(x_-) < 0$, then in an open interval centred at x_+ , $f(x)$ can be approximated upto the second order by $f(x) \cong f(x_\pm) + 9^{-1}2^{1-3\times 2^{-1}n}\left[f^{'}(x_\pm)\,\psi_{k,n}(n)\right] - 6\,\pi^{-2}f^{''}(x_\pm)[\varphi(x - x_\pm) - 1]$.

Practically in approximation problems, $f(x)$ and the point x_0 are given, and we need to approximate $f(x)$ up to 2nd order by equation given in above result. So to obtain the values of $n\,, k$ we fix one parameter and obtain the other by the equation $x_0 = 2^{-n}(k + 2^{-1} \pm 3^{-1})$. In fact , we take $k = \lfloor x_0 - (2^{-1} \pm 3^{-1})\rfloor$, $n = \lfloor log_2(\,(k + 2^{-1} \pm 3^{-1})(x_0)^{-1})\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Following theorem gives us an estimate of approximation error under some further restrictions on f.

10.2Theorem [60]: Let f be a given bounded function, such that, for fixed $n\,, k$, in one of the two points $x_\pm = 2^{-n}\left(k + (2^{-1} \pm 3^{-1})\right)$, it is at least C^2 and $f^{'}(x_+) > 0\,, f^{'}(x_-) < 0$ with $f(x) < K, x \in I_\pm = (x_\pm - \delta\,, x_\pm + \delta)$ for some $\delta > 0$. Then in $I_\pm$ the approximation error ,$\in_\pm$ satisfies $\in_\pm \le K + \left|\mp 9^{-1}2^{1-3\times 2^{-1}n}\left[f^{'}(x_\pm)\,\psi_{k,n}(x_\pm \mp \delta)\right] + 6\pi^{-2}f^{''}(x_\pm)[\varphi(\delta) - 1]\right|, x \in I_\pm$.

It is claimed in [60] that up to second order, the approximation presented in theorem 10.1 is more efficient than the representation given in (1)**.** Moreover**,** the approximation presented in theorem 10.1 can be used for a more general class of functions, C^2 functions although locally, while (1) is restricted only to the $\mathbb{L}^2(\mathbb{R})$ functions, in fact only for case when $f \in B_\psi$.

XI. Method of solving fractional differential equations by using Shannon wavelet analysis

Since fractional order differential equations appear in the modelling of real world problems occurring in many fields such as fluid mechanics, bio medical signal processing, engineering, computer science, physics etc, a lot of attention has been paid to formulate methods to find exact and numerical solutions of fractional differential equations. As analytic solutions are usually hard to obtain much effort has been paid to obtain methods which solve fractional differential equations numerically. For example, Gejji and Jafari in [61], Ray et al in [62] and Wang in [63] used Adominian Decomposition method; Momani and Odibat in [64], Nawaz in [65], and Obidat et al in [63] used variation Iteration method; Hosseinnia et al in [67] and Sweilam et al in [68] used Homotopy Perturbation method; Zurigat et al in [69] used Homotopy analysis method; Ervin and Roop in [70] and Saadatmandi and Dehghan in [71] used spectral methods and Chen and Wu in [72] used Haar wavelet method to solve fractional order differential equations. Also, Saeedi et al in [73] used CAS wavelet operational matrix of fractional order integration to solve integro- differential equation of fractional order.

Motivated by these works, Nouri and Siavashani [74] used Shannon wavelet analysis for solving boundary value problems of fractional differential equations. In fact, the authors derived a Shannon wavelet operational matrix of fractional order integration and applied it to find the solution of the Boundary value problems for fractional differential equation $D^\alpha y(t) = f\left(t, y(t), D^\beta(t)\right)$, $0 \le t \le 1$ with boundary conditions $y(0) = y_0$, $y(1) = y_1$ and $\alpha\,, \beta > 0$.

XII. Evaluation of radar cross section(RCS) by using Shannon wavelet analysis

Scientists have been engaged, for several decades to study scattering problems [27]. One of the most important parameters, among many, in the study of scattering is the electromagnetic scattering by a target which is traditionally represented by its RCS [27] . The RCS is defined as "the area intercepting the amount of power that, when scattered isotropically, produces at the receiver a density that is equal to the density scattered by the actual target" [75]. If the location of the transmitter and receiver is same, then RCS is usually known as "monostatic" otherwise RCS is known as "bistatic". In [27] , Danesfahani and Varmazyar used Shannon wavelet basis to the method of moments to find RCS of conductive and resistive surfaces. They used Fredholm integral equations of first and second kind to model the problem; afterwards they used the method, proposed by them and recorded by us in section 7, based on the "method of moments" and "Shannon wavelet basis ", to solve the obtained Fredholm integral equations and hence to evaluated the RCS .

XIII. Applications of Shannon wavelet analysis to study human DNA

Deoxyribonucleic acid (DNA) is a double helix consisting of two polymers which are connected by hydrogen atoms. The constituent polymers of DNA are three types of nucleotides: Deoxyribose, Phosphate group and Nitrogenous base. Nitrogenous bases are further of four types: Thymine (denoted by the symbol T), Adenine (denoted by the symbol A), Guanine (denoted by the symbol G) and Cytosine (denoted by the symbol C).The four bases are connected in such a way that one strand is connected with exactly one type of base on other strand, thus forming a pairing known as “base pairing”. In fact A is connected to T and C is connected to G. The genetic code is based on these four bases and they instruct the cells of the body, how to synthesis proteins and enzymes. On average each chromosome contains 160 million nucleotide pairs and there are 24 chromosomes in each cell. A lot of chromosomes data has been collected during recent years and is made available for scientific research. The available chromosomes data also includes a fifth symbol ‘N’ in addition to T, C, A and G, but it partically does not have role in DNA coding. While coding the symbols T, C, A, G, N, are being transformed to numerical values. But since N does not have any effect in coding and pairing happens in T, C, A, G, so while translating the symbols into numerical values (for analysis) care must be taken to reflect the base pairing restriction and the fifth symbol existence. Keeping these points in view, we follow the following symbol translation: $A = 1 + i0, C = -1 + i0, T = 0 + i1, G = 0 - i1, N = 0 + i0$, where $i = \sqrt{-1}$. Because of this numerical transformation of symbols, a sequence of numbers is obtained along the DNA strand and thus we get a signal $x(t)$. Usually the signal $x(t)$ is complicated and can’t be analysed in “time domain”. Thus to study various characteristics of $x(t)$, we study it in frequency domain by applying some transform on it. Machado et al [76] applied continuous wavelet transform by using followings six mother wavelets: Haar wavelet, Richer Wavelet, Shannon Wavelet, Hermition wavelet and Morlet Wavelet. They found that best results(with respect to interpretation and comparison) are obtained by using Shannon wavelets.

XIV. Application Of Shannon wavelet analysis to improve quality of science for telemedicine

In Intensive Care Unit (ICU) telemedicine, the most common form of telecommunication that has been used is “Broad band”. But generally Broadband, particularly in rural areas, has not been installed because of high cost on its networks infrastructure. So in rural areas, generally, lower bandwidth is installed for telecommunication in ICU telemedicine. As a result there is service contention at the customer’s place. Thus a challenging task is to have better service with a lower bandwidth. In their research work [77], the authors have used Shannon wavelets along with Daubechies wavelet to compress the data of ICU and thus achieving the goal of "Providing quality service (up to a certain level) using lower bandwidth in on access pipes."

XV. Applications of Shannon wavelet analysis to pathologic onion image segmentation

As it is a known fact that in the post harvesting process, one important step is to store onion for future use. But when the onion is stored, it gets infected with Pathogen due to pests, soak and over- nitrogen. This leads to rotting in the packages and hence onion loss. Thus it is very important to grade and classify onions, in the post harvesting, so that the loss of onions get reduce. For this manual grading and classification of onions is really a huge task and is also not too fruitful, thus technology is required for this purpose. In this direction Image measurement technology discussed by Chen et al [78] is a new method in addition to earlier existing methods. One of the important part of this image measurement technology is Image segmentation. There are several classical image segmentation methods such as Sobel, Canny, quadtree and OTSU algorithm. All these methods take the gradient of the image as the feature descriptor directly in image segmentation. But in these methods analysis of geometric properties of the target remains difficult to handle as the object boundary and target pixels, obtained by these methods, are often unclosed, thus econometric analysis of segmentation results is very difficult. So some new methods are required to this end. In this context wavelet precise integration method (WPIM) has been developed, in recent years, to solve non-linear partial differential equations for image processing which really has been proving to produce efficient and precise image processing.

In [79], Wang and Thei used Shannon wavelet in WPIM for pathologic onion image segmentation. After comparing they found that their method produced much better result in comparison to others. In their method, they firstly constructed a "Shannon wavelet interpolation scheme" by exploiting the interpolation property of Shannon wavelet and homotopy perturbation method and as collection points of WPIM they took image pixels of Burkholderia Capacia infected onions. Then they discretized the image segmentation model (C-V model) into a system of non-linear ordinary differential equation. They solved this system of ODEs by the half analytical scheme combined with HPM and the precision integration method. At the end of their work, they discussed and compared the numerical efficiency and precision of WPIM with other methods such as OSTU method and Sobel operator.

XVI. Applications to finance

[80] used Shannon wavelet analysis to value financial options. The technique that the authors developed is named as Shannon wavelet inverse Fourier technique(SWIFT). The exceptional nature of the local Shannon wavelets basis enabled the authors to adaptively determine the proper size of the computational interval. The authors climed that the SWIFT method will be applied to early-exercise options as well as in the context of risk management to compute the risk measures.

XVII. Conclusions

As said earlier, among the many families of wavelets available in the literature, Shannon wavelets offer some more specific advantages, which are usually missing in the others such as:
(1) Shannon wavelets are infinitely differentiable ;
(2) Shannon wavelets are analytically defined;
(3) Shannon wavelets are shapely bounded in the frequency domain;
(4) Shannon wavelets enjoy a generalization of the Shannon sampling theorem;
(5) Shannon wavelets give rise to the connection coefficients which can be analytically defined for any order derivatives, while for the other wavelet families they are computed only numerically that too for the lower order derivatives only.

Keeping this in mind it is clear that Shannon wavelets have huge potential to get applied to solve differential, integral and real world problems. Although good amount of work has been done in this direction, yet in comparison to the Haar wavelet [17] a very small amount of work has been done. So for researchers there is a huge opportunity to work with Shannon wavelets extensively, which may provide efficient methods of handling differential, integral and real world problems, as we have already seen in this survey in many cases where Shannon wavelets have been utilised.

References

[1]. Bachmana, N.L. and Beckenstein , E., (2000), *Fourier and wavelet Analysis* (springer - verlay New york , Inc.)

[2]. Cooley , J., Tukey , J . (1965) An algorithm for the machine calculation of complex Fourier series , *Math . Com . 19* , 297-301 .

[3]. Debnath, L., (2002), *Wavelet Transforms and their Applications* (Birkhauser.)

[4]. Hong, D. Wang , J. and Gardner , R . (2005), (*Real Analyisis with an introduction to wavelets and Applications* (Academic Press.)

[5]. Cattani, C.,(2006), Connection coefficients of Shannon wavelets, *Mathematical Modelling and Analysis, vol. 11, no. 2*, pp. 117–132.

[6]. Cattani, C., (2012), Fractional Calculus and Shannon wavelet, *Mathematical Problems in Engineering , Vol . 2012*, Article ID 502812,26 Pages.

[7]. Cattani, C.,(2008), Shannon wavelets theory, *Mathematical Problems in Engineering, vol. 2008,* Article ID164808, 24 pages.

[8]. Cattani, C.,(2010), Shannon wavelets for the solution of integro differential equations, *Mathematical Problems in Engineering, vol. 2010*, Article ID 408418, 22 pages.

[9]. Shi, Z. and Li, F.,(2014), Numerical solution of high order differential equations by using periodized Shannon wavelets, *Applied Mathematical modeling, 38(2014)* ,2235-2248.

[10]. Maleknajad, K. and Attary, M.,(2011),An efficient numerical approximation for the linear class of Fredholm integro-differential equations based on Cattani's method, *Commun Non Sci Numer Simulat, 16,* 2672-2679.

[11]. Maleknejad, K., Mahmoudi,Y.,(2003), Numerical solution of integro-differential equation by using Hybrid Taylor and Block-Pulse functions. *J Math Sci;9(2)*:203–13.

[12]. Rashidinia, J., Zarebnia, M., (2007),. The numerical solution of integro-differential equation by means of the Sinc method. *Appl Math Comput ;188*:1124–30.

[13]. Beylkin, G., Coifman, R. and Rokhlin, V., (1991), Fast wavelet transforms and numerical algorithms I, *Commun. Pure Appl. Math., 44,* 141—183.

[14]. Babolian, E., Shahsavaran, A., (2009), Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, *J. Com. Appl. Math.* 225 87 95.

[15]. Chen, W.S. and Lin, W., (2001), Galerkin trigonometric wavelet methods for the natural boundary integral equations, *Appl. Math. Comput. 121* 79—92.

[16]. Goswami, J.C., Chan, A.K. and Chui, C.K., (1995), On solving first-kind integral equations using wavelets on a bounded interval. *IEEE Trans.Antennas and Propagation, 43 (6)*: 614-622.

[17]. Hariharan, G. and Kannan, K. (2013), An overview of Haar method for wavelet solving Differential and Integral equations, *World Applied Science Journal 23(12)*:01-14.

[18]. Hsiao, G. C. and Rathsfeld, A., (2002), Wavelet collocation methods for a first kind boundary integral equation in acoustic scattering, *Adv. Comput. Math., 17* 281—308

[19]. Lage, C., Schwab, C., (1999), Wavelet Galerkin algorithms for boundary integral equations, *SIAM J. Sci. Comp., 20* 2195—2222.

[20]. Lepik, U. and Teemme, E., (2007), Solution of non-linear Fredholm Integral equations via Haar wavelet method; *proc. Estonian Acad. Sci. Phy. Math., 56,1*,17-27.

[21]. Lepik, U. and Tamme, E., (2005), Application of the Haar wavelets for solution of linear integral equations, in: *Dynamical systems and applications*, pp. 395-407.

[22]. Lin, Y., Leng, S. and Lu, Y., (2009), A Haar wavelet solution to Fredholm equations. *IEEE Transactions, 978-1-4244-4507*-3/09.

[23]. Mahmoudi, Y., (2005), Wavelet Galerkin method for numerical solution of nonlinear integral equation, *Appl. Math. Comput. 167* 1119_1129.

[24]. Maleknejad, K. and Mirzaee, F., (2005), Using rationalized Haar wavelet for solving linear integral equations. *Appl. Math. Comput. 160 (2)*:579-587.

[25]. Maleknejad, K., Karami, M., (2005), Numerical solution of non-linear Fredholm integral equations by using multi wavelet in Petrov-Galerkin method, *Appl. Math. Comput. 168* 102_110.

[26]. Shamsi, M., Razzaghi, M., Nazarzadeh, J. and Shafiee, M., (2004), Haar wavelet method for solving Pocklington's integral equation. Kybernetika, *JO, 4*: 491-500.

[27]. Danesfahani, R., Hatamzadeh-varmazyar, S., Babolian, E. and Masouri, Z.,(2008), Applying Shannon wavelet basis functions to the method of moments for evaluating the Radar cross-section of the conducting and resistive surfaces, Progress in electromagnetic research B, *Vol.8*,257-292.

[28]. Siddiqi, S. S, Akram, G. and Zaheer, S., (2009), Solution of tenth order boundary value problems using variational iteration technique, *EuropeanJournal of Scientific Research 30* 326–347.

[29]. Siddiqi ,S. S, Akram, G, Zulfiqar,I., (2009), Solution of eleventh order boundary value problems using variational iteration technique, *EuropeanJournal of Scientific Research 30* 505–525.

[30]. Siddiqi ,S. S, Akram, G, Zulfiqar., (2009), Solution of twelfth order boundary value problems using variational iteration technique, *European Journal of Scientific Research 33* 96–114.

[31]. Siddiqi, S. S, Akram, G., (2007), Solution of 10th-order boundary value problems using non-polynomial spline technique, *Applied Mathematics andComputation 190* 641–651.

[32]. Mai-Duy ,N., (2006), An effective spectral collocation method for the direct solution of high-order ODEs, *Communications in Numerical Methods in Engineering22* 627–642.

[33]. Mai-Duy ,N., Tanner, R.I., (2005), Solving high-order partial differential equations with indirect radial basis function networks, *International Journal forNumerical Methods in Engineering 63* 1636–1654.

[34]. Reginska, T., Elden, L.,(1997); Solving the sideways heat equation by a wavelet-Galerkin method, *Inverse Problems, 13:* 1093-1106.

[35]. Wang, J.R.,(2005), The multi-resolution method applied to the sideways heat equation, *Journal of Mathematical Analysis and Applications, vol. 309, no. 2*, pp. 661–673.

[36]. Seidman, T., Elden,L.,(1990); An optimal iterating method for the sideways heat equation. *Inverse Problems, 6*: 681-696.

[37]. Beck,J.V.,(1970); Nonlinear estimation applied to the nonlinear inverse heat conduction problem, *Int. J. Heat Mass Transfer, 13*: 703-716.

[38]. Qian, Z., Fu,C.L., Xiong,X.T.,(2007); A modified method for determining the surface sheat ux of IHCP, *Inverse Probl. Sci. Eng., 15(3)*: 249-265.

[39]. Carasso, A., (1982), Determining surface temperatures from interior observations, *SIAM Journal on Applied Mathematics, vol. 42, no. 3*, pp. 558–574.

[40]. Fu,C.L., (2004),Simplified Tikhonov and Fourier regularization methods on a general sideways parabolic equation, *Journal of Computational and Applied Mathematics, vol. 167, no. 2*, pp. 449–463.

[41]. Murio, D.A.,(1993), The Mollification Method and the Numerical Solution of ill-Posed Problems, A Wiley- Interscience Publication, JohnWiley & Sons, New York, NY, USA.

[42]. Hao, D.N. and Reinhardt, H.J.(1997), On a sideways parabolic equation, *Inverse Problems, vol. 13, no. 2*, pp. 297–309.

[43]. Elden, L., Berntsson, F. and Reginska, T.,(2000), Wavelet and Fourier methods for solving the sideways heat equation, *SIAM Journal on Scientific Computing, vol. 21, no. 6*, pp. 2187–2205.

[44]. Reginska, T. and Elden, L.,(1997), Solving the sideways heat equation by a wavelet-Galerkin method, *Inverse Problems, vol. 13, no. 4*, pp. 1093–1106.

[45]. Reginska, T. and Elden, L.,(2000), Stability and convergence of the wavelet-Galerkin method for the sideways heat equation, *Journal of Inverse and ill-Posed Problems, vol. 8, no. 1*, pp. 31–49.

[46]. Fu, C.L. and Qiu, C.Y.,(2003), Wavelet and error estimation of surface heat flux, *Journal of Computational and Applied Mathematics, vol. 150, no. 1*, pp. 143–155.

[47]. Xiong, X.T. and Fu, C.L.,(2007), Determining surface temperature and heat flux by a wavelet dual least squares method, *Journal of Computational and Applied Mathematics, vol. 201, no. 1*, pp. 198–207.

[48]. Cheng,W., Fu,C.L. and Qian,Z.,(2007), A modified Tikhonov regularization method for a spherically symmetric three-dimensional inverse heat conduction problem, *Mathematics and Computers in Simulation, vol. 75, no. 3-4*, pp. 97–112.

[49]. Tautenhahn, U.,(1997), Optimal stable approximations for the sideways heat equation, *Journal of Inverse and ill-Posed Problems, vol. 5, no. 3*, pp. 287–307.

[50]. Chen, H. T., Lin, S. Y. and Fang, L. C.,(2001), Estimation of surface temperature in two-dimensional inverse heat conduction problems, *International Journal of Heat and Mass Transfer, vol. 44, no. 8*, pp. 1455–1463.

[51]. Busby, H,R. and Trujillo, D.M.,(1985), Numerical solution to a two-dimensional inverse heat conduction problem, *International Journal for Numerical Methods in Engineering, vol. 21, no. 2*, pp. 349–359.

[52]. Alifanov,O.M. and Kerov,N.V.,(1981), Determination of external thermal load parameters by solving the two-dimensional inverse heat-conduction problem, *Journal of Engineering Physics, vol. 41, no. 4*, pp. 1049–1053.

[53]. Gualous, H.L., Panday, P.K., and Artyukhin, E.A.,(2003), The inverse determination of the local heat transfer coefficients for nucleate boiling on horizontal cylinder, *Journal of Heat Transfer, vol. 125, no. 1*, pp. 1087–1095.

[54]. Xiong, X.T., Fu, C.L., Li, H.F.,(2004); Central difference schemes in time and error estimate on a nonstandard inverse heat conduction problem, *Appl. Math. Comput., 157(1)*: 77-91.

[55]. Xiong, X.T., Fu, C.L., Li, H.F.,(2006); Central difference method of a non-standard inverse heat conduction problem for determining surface heat flux from interior observations, *Appl. Math. Comput., 173*: 1265-1287.

[56]. Reginska, T.,(2001), Application of wavelet shrinkage to solving the sideways heat equation, *BIT Numerical Mathematics, vol. 41, no. 5,* pp. 1101–1110.

[57]. Cheng, W., Zhang, Y-Qi. and Fu,C,Li.,(2013), A wavelet regularization method for an inverse heat conduction problem with convection terms, *Electronic journal of differential equations,Vol(2013),No. 122,*PP.1-9.

[58]. Fu, C.L.,(2004); Simplified Tikhonov and Fourier regularization methods on a general sideways parabolic equation, *Comput. Appl. Math., 167*: 449-463.

[59]. Nair,M.T., Tautenhahn,U.,(2004); Lavrentiev regularization for linear ill-posed problems under general source conditions, *J. Anal. Appl., 23(1)*: 167-185.

[60]. Cattani, C.,(2011), Second order Shannon wavelet approximation of C^2-functions, *U.P.B. Sci.Bull.,Series A,Vol.73,Iss.3*.

[61]. Gejji, V.D., Jafari, H.,(2007), Solving a multi-order fractional differential equation, *Appl. Math. Comput., 189* , 541-548.

[62]. Ray, S.S., Chaudhuri, K.S., Bera, R.K.,(2006), Analytical approximate solution of nonlinear dynamic system containing fractional derivative by modified decomposition method, *Appl. Math. Comput., 182,* 544-552.

[63]. Wang, Q.,(2006), Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method, *Appl. Math. Comput., 182* ,1048-1055.

[64]. Momani, S., Odibat, Z.,(2006), Analytical approach to linear fractional partial differential equations arising in fluid mechanics, *Phys. Lett. A, 355*, 271-279.

[65]. Nawaz, Y., (2011), Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations, *Comput.Math. Appl., 61* , 2330-2341.

[66]. Odibat, Z., Momani, S.,(2009), The variational iteration method: an efficient scheme for handling fractional partial differential equations in fluidmechanics, *Comput. Math. Appl., 58 ,* 2199-2208.

[67]. Hosseinnia, S., Ranjbar, A., Momani, S.,(2008), Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part, Comput. Math. Appl., 56 , 3138-3149.

[68]. Sweilam, N.H., Khader, M.M., Al-Bar, R.F.,(2007), Numerical studies for a multi-order fractional differential equation, *Phys. Lett. A, 371* ,26-33.

[69]. Zurigat, M., Momani, S., Alawneh, A.,(2010), Analytical approximate solutions of systems of fractional algebraic-differential equations by homotopy analysis method, *Comput. Math. Appl., 59*, 1227-1235.

[70]. Ervin, V.J., Roop, J.P.,(2006), Variational formulation for the stationary fractional advection dispersion equation, Number. *Methods Partial Differential Equations, 22* , 558-576.

[71]. Saadatmandi., Dehghan, M.,(2011), A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl., 59* (2010),1326-1336.

[72]. Chen, Y.M.,Wu, Y.B.,(2010), Wavelet method for a class of fractional convection-diffusion equation with variable coefficients, J. Comput. Sci., 1, 146-149.

[73]. Saeedi, H., Moghadam, M., M., Mollahasani, N., Chuev, G.N.,(2011), A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order, *Commun. Nonlinear Sci. Numer. Simul., 16,* 1154-1163.

[74]. Nouri,K. and Siavashani, N. B., (2014), "Application of Shannon wavelet for solving boundary value problems of fractional differential equations", *Wavelets and Linear Algebra, 1,* 25-32,

[75]. Balanis, C. A., (1982), *Antenna Theory*: *Analysis and Design* (Wiley,New York.)

[76]. Mochado, J.A.T, Costa, A.C. and Quelhas, M.D.(2011), "Wavelet Analysis of human DNA", *Genomics, volume 98, Issue 3,* Pages 155 -163.

[77]. Salatian, A., Adepoju,F .and Oborkhale,L.(2011),Using wavelets to improve Quality of service for Telemedicine, *International Journal of Advanced Science and Technology ,Vol.27*.

[78]. Chen, Y., Xia, Y., Bian, Y. and Zhang, Z.P.,(2010) " Image measurement of precision aluminium alloy forgings" , *Journal of Plasticity Engineering, Vol 17, no.6,* PP. 77-81.

[79]. Wang, H. and Mei, S.(2014), Shannon Wavelet Precision Integration Method for Pathologic Onion Image Segmentation Based on Homotopy Perturbation Technology, *Hindawi Publishing Corporation, Mathematical Problems in Engineering, Volume 2014, article ID 601841*, 10 pages.

[80]. Gracia , L. and Oosterleec, W. (2016) *"*A highly efficient Shannon wavelet inverse Fourier technique for pricing European options", *SIAM J. SCI. COMPUT. Vol. 38, No. 1,* pp. B118–B143