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Fixed point theorems in Banach spaces using three steps iteration

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Abstract: *Suppose K is a nonempty closed convex nonexpansive retract of real uniformly convex Banach space E with P as a nonexpansive retraction. Let T: K → E be a nonexpansive non-self map with* $F(T)=\{x\in K: Tx=x\}\neq\phi$. *suppose* $\{x_n\}$ *is generated iteratively by*

$$x_1\in K, x_{n+1}=P((1-\alpha_n)x_n+\alpha_n TP[(1-\beta_n)x_n+\beta_n Tx_n]),$$

$n\geq 1$ *Where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon\in(0,1)$. *(1) If the dual* E^* *of E has the Kadec-Klee property, then weak convergence of* $\{x_n\}$ *to some* $x^*\in F(T)$ *is proved. (2) If T satisfies condition (A), then strong convergence of* $\{x_n\}$ *to some* $x^*\in F(T)$ *is obtained.*

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I. Introduction

Let K be a nonempty subset of a real normed linear space E. Let T be a self-mapping of K. Then T is said to be nonexpansive if

$$\|Tx-Ty\|\leq\|x-y\| \tag{1.1}$$

For all x, y ∈ K

Definition 1.1:
Let X be a real Banach space. A subset K of X is said to be a retract of X if there exists a continuous map P: X→K such that Px = x for all x ∈ K. A map P: X → X is said to be a retraction if $P^2 = P$. It follows that if P is real retraction map, then Py = y for all y in the range of P.

Let K be a monempty convex subset of X and T: K→ K. For $x_i \in K$ and some $\{\alpha_n\}$, $\{\beta_n\}$ $\{\gamma_n\}\subseteq[0,1]$.

The Mann iteration formula is given as

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n Tx_n \qquad n\geq 1$$

The Ishikawa iterative scheme is defined by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n$$
$$y_n=(1-\beta_n)x_n+\beta_n Tx_n \qquad n\geq 1$$

The three step Noor iteration scheme is defined by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n$$
$$y_n=(1-\beta_n)x_n+\beta_n Tz_n$$
$$z_n=(1-\gamma_n)x_n+\gamma_n Tx_n$$

Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see e.g. [5, 13, 15, 17] using the Mann iteration method (see e.g. [10]) or the Ishikawa iteration method (see e.g. [6]) and Noor [18].

In [15], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive mapping defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. More precisely, they proved the following theorem.

Theorem XU (Tan and Xu[15, Theorem1, p. 305] Let E be a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm and K a monempty closed convex bounded subset of E.

Let T: K→ K be a nonexpansive mapping. Let T: K→ K be a nonexpansive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0, 1], such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty, \sum_{n=1}^{\infty}\beta_n(1-\alpha_n)<\infty$ and lim $\sup_{n\to\infty}\beta_n<1.$ Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T[(1-\beta_n)x_n+\beta_n Tx_n]), n\geq 1 \quad (1.2)$$

Converges weakly to some fixed point of T.

In the above result, T remains self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain K of T is a proper subset of E (and this is the case in several applications), and T maps K into E then, the iteration formula (1.2) may fail to be well defined.

The purpose of this paper is to construct an iteration scheme for approximation a fixed point of nonexpansive non-self maps (when such a fixed point exists) and to prove some strong and weak convergence theorems for such maps. Our theorems improve and generalize some previous results. Our weak convergence result applies not only to L^p- spaces with $1<p<\infty$ but also to other spaces which do not satisfy Opial's condition or have a Frechet differentiable norm. More precisely, we prove weak convergence of the modified Ishikawa-type iteration process in a uniformly convex Banach space whose dual has the Kadec- Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Frechat differentiable norm nor Opial property; however, their dual does have the Kadec-Klee property (see, e.g.,[4,7]).

II. Preliminaries

Let E be a real Banach space. A subset K of E is said to b a retrct of E if there exists a continuous map P: E → K such that P_x = x for all $x \in K$. A map P: E→ E is said to be retraction if $P^2 = P$. If follows that if a map P is a retraction, then Py = y for all y in the range of P. A set K is optimal if each point outside K can be moved to be closer to all points of K. It is well know (e.g., [2]) that

(1) If E is a separable, strictly convex, smooth, reflexive Banach space and if $K\subset E$ an optimal set with interior, then K is is a nonexpansive rectrct of E.

(2) A subset of l_p, with $1<p<\infty$, is a nonexpansive tetract if and only if it is optiomal.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and combex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retracts.

A mapping T with domain D(T) and range R(T) in E is said to be semiclosed at p if whenever $\{x_n\}$is a sequence in D(T) such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strogly to p, then $Tx^* = p$.

A Banach space E is said to have the Kadec-Klee property if for every sequences $\{x_n\}$ in E, $x_n\to x$ weakly and $\|x_n\|\to\|x\|$ strongly together imply $\|x_n-x\|\to 0.$

In what follows, we shall make use of the following lemmas:

Lemma2.1 (Schu [12]) Let E be a uniformly convex Banach spaces and $\{\alpha_n\}$ a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon\in(0,1)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty}\|x_n\|\leq r, \limsup_{n\to\infty}\|y_n\|\leq r, and \limsup_{n\to\infty}\|\alpha_n x+(1-\alpha_n)y_n\|=r,$ hold for some r ≥ 0. Then $\lim_{n\to\infty}\|x_n-y_n\|=0$

Lemma 2.2 (Xu [16]) Let p >1 and R > 1 be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continous, strictly increasing and convex function g: [0, ∞) → [0, ∞) with g(0) = 0 such that $\|\lambda x+(1-\lambda)y\|^p\leq\lambda\|x^p\|+(1-\lambda)\|y\ \|^p-W_p(\lambda)_g(\|x-y\|)$

For all x, y $\in B_R(0)=\{x\in E:\|x\|\leq R\}, and \lambda\in[0,1],$ where $W_p(\lambda)=\lambda(1-\lambda)^p+\lambda^p(1-\lambda).$

Lemma 2.3 (Tan and Xu [15]) Let $\{\lambda_n\}$and $\{\sigma_n\}$be sequences of nonnegative real numbers such that $\lambda_{n+1}\leq\lambda_n+\sigma_n, \forall n\geq 1$and $\sum_{n=1}^{\infty}\sigma_n<\infty.$ Then $\lim_{n\to\infty}\lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{nj}\}\to 0$ as $j\to\infty$, then $\lambda_n\to 0$ as $n\to\infty.$

Lemma 2.4 (Kaczor [7] Let E be a real reflexive Banach space such that it's dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and x^*, $y^* \in \omega_\omega(x_n)$; here $\omega_\omega(x_n)$ denotes the weak w-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty}\|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.

Lemma 2.5 (Browder [1]) Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E and T: K→ E a nonexpansive mapping. Then I- T is semi closed at zero.

III. Main Results

In this section, we prove our main theorems.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E, which is also a nonexpansive retract of E. Let T: K → E be a nonexpansive mapping.

The following iteration scheme is studied:

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n] + \beta_n TP[(1-\gamma_n)x_n + \gamma_n Tx_n]), \qquad (3.1)$$

With $x_1 \in K, n \ge 1$ where $\{\alpha_n\}$ $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0, 1] and P is a nonexpansive retraction of E onto K.

Theorem 3.1 Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let T: K→ E, be a nonexpansive mapping with $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Then $\lim_{n\to\infty}\|x_n - x^*\|$ exits.

Proof: We observe that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P((1-\alpha_n)x_n + \alpha_n Ty_n) - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|P((1-\beta_n)x_n + \beta_n Tz_n) - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\|Tz_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\|z_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x_n - x^*\| + \alpha_n\beta_n(1-\gamma_n)\|x_n - x^*\| + \alpha_n\beta_n\gamma_n\|x_n - x^*\| \\
&= \|x_n - x\|
\end{aligned}$$

Consequently, we have

$$\|x_{n+1} - x^*\| \le \|x_1 - x^*\|.$$

This implies that $\{x_n\}$ is bounded and Lemma 2.3 guarantees that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

This completes the proof.

Theorem 3.2 Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is a also a nonexpansive retract of E. Let T: K → E be a nonexpansive mapping with $F(T) \ne \phi$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Then

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0$$

Proof: Let $x^* \in F(T)$. then, by theorm 3.1, $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ exists. Let $\lim_{n\to\infty}\|x_n - x^*\| = 0$. If r = 0, Then by the continuity of T the conclusion follows. Now suppose r > 0. We claim

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0$$

Now set $y_n = P[(1-\beta_n)x_n + \beta_n Tz_n]$. Since $\{x_n\}$ is bounded, There exists R > 0 such that $x_n - x^*$, $y_n - x^*$ and $z_n - x^* \in B_R(0)$ for all $n \ge 1$. Using Lemma 2.2, we have that

$$\|z_n - x^*\|^2 = \|P((1-\gamma_n)x_n + \gamma_n Tx_n) - x^*\|^2$$
$$= \|P(1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n)x^* - \gamma_n x^*\|^2$$
$$\leq (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2$$
$$\leq (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 - W_2(\gamma_n)g(\|Tx_n - x_n\|)$$
$$\leq \gamma_n\|x_n - x^*\|^2 + (1-\lambda_n)\|x_n - x^*\|^2$$
\
$$= \|x_n - x^*\|^2$$

From Lemma 2.2, it follows that

$$\|x_{n+1} - x^*\|^2 = \|P[(1-\alpha_n)x_n + \alpha_n Ty_n] - Px^*\|^2$$
$$\leq \|\alpha_n(Ty_n - x^*) + (1-\alpha_n)(x_n - x^*)\|^2$$
$$\leq \alpha_n\|y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Ty_n - x_n\|)$$
$$\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Ty_n - x_n\|)$$
$$\leq \|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Ty_n - x_n\|). \quad (3.2)$$

Observe that $W_2(\alpha_n) \geq \varepsilon^3$. Now (3.2) implies that

$$\varepsilon^3 \sum_{n=1}^{\infty} g(\|Tz_n - x_n\|) \leq \|x_1 - x^*\|^2 < \infty$$

Therefore, we have $\lim_{n\to\infty} g(\|Tz_n - x_n\|) = 0.$ since g is strictly increasing continuous at 0. It follows that

$$\lim_{n\to\infty}\|Tz_n - x_n\| = 0.$$

Since T is nonexpansive, we can get that

$$\|z_n - x^*\| \leq \|z_n - Ty_n\| + \|y_n - x^*\|$$

Which on taking $\lim\inf_{n\to\infty}$ gives $r \leq \lim\inf_{n\to\infty}\|z_n - x^*\|$. On the other hand, we have

$$\|z_n - x^*\| \leq \|(1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n + \gamma_n)x^*\|$$
$$\leq \|\gamma_n(x_n - x^*) + (1-\gamma_n)(x_n - x^*\|$$
$$\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\|$$
$$= \|x_n - x^*\|$$

Which implies $\lim\sup_{n\to\infty}\|z_n - x^*\| \leq r$. Therefore, $\lim_{n\to\infty}\|z_n - x^*\| = r$ and so

$$\lim_{n\to\infty}\|\beta_n(Tz_n - x^*) + (1-\gamma_n)(z_n - x^*)\| = r$$

Since $\lim\sup_{n\to\infty}\|Tx_n - x^*\| \leq r$, it fallows from Lemma 2.1 that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

This completes the proof.

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