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Sampling Expansion with Symmetric Multi-Channel Sampling in a series of Shift-Invariant Spaces

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Abstract: *We find necessary and sufficient conditions under which a regular shifted sampling expansion hold on $\sum_{d=1}^{m} V\ (\varphi(t_d))$ and obtain truncation error estimates of the sampling series. We also find a sufficient condition for a function in $L^2(\mathbb{R})$ that belongs to a sampling subspace of $L^2(\mathbb{R})$. We use Fourier duality between $\sum_{d=1}^{m} V\ (\varphi(t_d))$ and $L^2[0,2\pi]$ to find conditions under which there is a stable asymmetric multi-channel sampling formula on$\sum_{d=1}^{m} V\ (\varphi(t_d))$.*

Keywords: *Shift invariant space, sampling expansion, Multi-channel sampling , Frame Riesz basis.*

I. Introduction

Let $\sum_{d=1}^{m}\varphi(t_d)$ in $L^2(\mathbb{R})$, let $\sum_{d=1}^{m} V\ (\varphi(t_d))$ = span$\{\sum_{d=1}^{m}\varphi(t_d-n) : n \in \mathbb{Z}\}$ be the closed subspace of $L^2(\mathbb{R})$spanned by integer translates $\{\sum_{d=1}^{m}\varphi(t_d-n) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m}\varphi(t_d)$. We call $\sum_{d=1}^{m} V\ (\varphi(t_d))$ the series of shift invariant space generated by $\sum_{d=1}^{m}\varphi(t_d)$ and $\sum_{d=1}^{m}\varphi(t_d)$ a frame or a Riesz or an orthonormal generator if $\{\sum_{d=1}^{m}\varphi(t_d-n) : n \in \mathbb{Z}\}$ is a frame or a Riesz basis or an orthonormal basis of $\sum_{d=1}^{m} V\ (\varphi(t_d))$. The multi-channel sampling method goes back to the works of Shannon [16] and Fogel [15], where reconstruction of a band-limited signal from samples of the signal and its derivatives was found. Generalized sampling expansion using arbitrary multi-channel sampling on the Paley–Wiener space was introduced first by Papoulis [14] .

Adam zakria , Ahmed Abdallatif ' Yousif Abdeltuif [1] and S. Kang , J.M. Kim, K.H. Kwon [12] considered sampling expansion in a series of shift invariant spaces and symmetric multi-channel sampling in shift-invariant spaces space $V\ (\varphi)$with a suitable Riesz generator $\varphi(t)$, where each channeled signal is sampled with a uniform but distinct rate.Using Fourier duality between $\sum_{d=1}^{m} V\ (\varphi(t_d))$ and $L^2[0,2\pi]$ [7,8,9,12], we derive under the same considerations a stable series of shifted asymmetric multi-channel sampling formula in $\sum_{d=1}^{m} V\ (\varphi(t_d))$. For example, Walter considered a real-valued continuous orthonormal generator satisfying $\sum_{d=1}^{m}\varphi(t_d) = O((1+\sum_{d=1}^{m}|t_d|)^{-s})$ with $s > 1$, Chen, Itoh, and Shiki considered a continuous Riesz generator satisfying $\sum_{d=1}^{m}\varphi(t_d) = O((1+\sum_{d=1}^{m}|t_d|)^{-s}$ with $s > \frac{1}{2}$, and Zhou and Sun considered a continuous frame generator $\sum_{d=1}^{m}\varphi(t_d)$ satisfying $sup_{\mathbb{R}}\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}|\varphi\ (t_d-n)|^2 < \infty$. We find necessary and sufficient conditions under which an irregular sampling expansion and a regular shifted sampling expansion hold on $\sum_{d=1}^{m} V(\varphi(t_d))$. We give an illustrative examples (see[6, 12]).

II. Preliminaries

We consider the notations and formulas in [6, 12]. Take $\{\varphi\ n : n \in \mathbb{Z}\}$ be a sequence of elements of a separable Hilbert space H with the inner product $(\ ,\ )$ and $V = \overline{span}\{\varphi\ n : n \in \mathbb{Z}\}$ the closed subspace of H spanned by $\{\varphi_n : n \in \mathbb{Z}\}$. Then $\{\varphi_n : n \in \mathbb{Z}\}$ is called

- a Bessel sequence (with a Bessel bound B) if there is a constant $A+\varepsilon_0 > 0$ such that $\sum_{n\in\mathbb{Z}}|\langle\varphi,\varphi_n\rangle|^2 \le (A+\varepsilon_0)||\ \varphi\ ||^2, \varphi \in H$ (or equivalently $\varphi \in V$),
- a frame sequence (with frame bounds $(A, A+\varepsilon_0)$) if there are constants $A, A+\varepsilon_0 > 0$ such that A$||\ \varphi\ ||^2 \le \sum_{n\in\mathbb{Z}}|\langle\varphi,\varphi_n\rangle|^2 \le (A+\varepsilon_0)||\ \varphi\ ||^2$, $\varphi \in V$, a Riesz sequence (with Riesz bounds $(A, A+\varepsilon_0)$) if there are constants $A+\varepsilon_0, A > 0$

$$A\|c\|^2 \le \left\|\sum_{n\in\mathbb{Z}} c\ (n)\ \varphi_n\right\|^2 \le (A+\varepsilon_0)\|c\|^2, c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$$

where $\|c\|^2 = \sum_{n\in\mathbb{Z}} |c(n)|^2$, an orthonormal sequence if $(\varphi_m,\varphi_n) = \delta_{m,n}$ for all m and n in $\mathbb{Z}$.

If $\{ \varphi_n : n \in \mathbb{Z}\}$ is a frame sequence or a Riesz sequence or an orthonormal sequence in H, then we say that $\{ \varphi_n : n \in \mathbb{Z}\}$ is a frame or a Riesz basis or an orthonormal basis of the Hilbert subspace V in H. On $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we take the Fourier transform to be normalized as

$$\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{-it\xi}\, dt, \varphi(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

so that $\mathcal{F}[\cdot]$ becomes a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

For any $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$, let $\sum_{d=1}^{m} \Phi(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi(t_d - n)|^2$,
$G_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\varphi(\xi + 2n\pi)|^2$. Then $\Phi(t) = \Phi(t+1) \in L^1[0,1]$,
$G_\varphi(\xi) = G_\varphi(\xi + 2\pi) \in L^1[0,2\pi]$ and
$|| \sum_{d=1}^{m} \varphi(t_d) ||^2_{L^2(\mathbb{R})} = || \sum_{d=1}^{m} \Phi(t_d) ||_{L^1[0,1]} = ||G_\varphi(\xi)||_{L^1[0,1]}$.

The normalized Fourier transform is

$$\mathcal{F}[\varphi](\xi) = \hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \sum_{d=1}^{m} \varphi(t_d) \prod_{d=1}^{m} e^{-it_d\xi}\, dt_d, \sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$$

so that $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\cdot]$ extends to a unitary operator from $L^2(\mathbb{R})$onto $L^2(\mathbb{R})$. For each $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$, let

$$\sum_{d=1}^{m} C_\varphi(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi(t_d + n)|^2 \text{ and } G_\varphi(\xi) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\xi + 2n\pi)|^2.$$

Hence

$$\sum_{d=1}^{m} C_\varphi(t_d) = \sum_{d=1}^{m} C_\varphi(t_d + 1) \in L^1[0,1], G_\varphi(\xi) = G_\varphi(\xi + 2\pi) \in L^2[0,2\pi]$$

and

$$\left\| \sum_{d=1}^{m} \varphi(t_d) \right\|^2_{L^2(\mathbb{R})} = \left\| \sum_{d=1}^{m} C_\varphi(t_d) \right\|_{L^1[0,1]} = \frac{1}{2\pi} \left\| G_\varphi(\xi) \right\|_{L^1[0,2\pi]}.$$

In particular, $\sum_{d=1}^{m} C_\varphi(t_d) < \infty$ for a.e. $\sum_{d=1}^{m} t_d \in \mathbb{R}$. We also let

$$\sum_{d=1}^{m} Z_\varphi(t_d, \xi) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} \varphi(t_d + n) e^{-in\xi}$$

be the Zak transform [11] of $\sum_{d=1}^{m} \varphi(t_d)$ in $L^2(\mathbb{R})$). Then $\sum_{d=1}^{m} Z_\varphi(t_d, \xi)$ is well defined a.e. on $\mathbb{R}^2$ and is quasi-periodic in the sense that
$\sum_{d=1}^{m} Z_\varphi(t_d + 1, \xi) = e^{i\xi} \sum_{d=1}^{m} Z_\varphi(t_d, \xi)$and $\sum_{d=1}^{m} Z_\varphi(t_d, \xi + 2\pi) = \sum_{d=1}^{m} Z_\varphi(t_d, \xi)$.

A Hilbert space H consisting of complex valued functions on a set E is called a reproducing kernel Hilbert space (RKHS in short) if there is a series of a functions $\sum_{d=1}^{m} q(s, t_d)$ on $E \times E$, called the reproducing kernel of H , satisfying
(i) $\sum_{d=1}^{m} q(., t_d) \in H$ for each $\sum_{d=1}^{m} t_d \in E$,
(ii) $\langle f(s), \sum_{d=1}^{m} q(s, t_d)\rangle = \sum_{d=1}^{m} f(t_d), f \in H$.

In an RKHS H, any norm converging sequence also converges uniformly on any subset of E, on which $\| \sum_{d=1}^{m} q(., t_d) \|^2_H = \sum_{d=1}^{m} q(t_d, t_d)$ is bounded.

A sequence $\{\varphi_n : n \in \mathbb{Z}\}$ of vectors in a separable Hilbert space H is
(i) a Bessel sequence with a bound $A + \varepsilon_0 : \varepsilon_0 > 0$ if

$$\sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \le (A + \varepsilon_0)\|\varphi\|^2, \varphi \in H, \varepsilon_0 > 0,$$

(ii) a frame of H with bounds $A + \varepsilon_0 \ge A : \varepsilon_0 > 0$ if

$$A\|\varphi\|^2 \le \sum_{n\in\mathbb{Z}} |\langle \varphi, \varphi_n \rangle|^2 \le (A + \varepsilon_0)\|\varphi\|^2, \varphi \in H, \varepsilon_0 > 0,$$

(iii) a Riesz basis of H with bounds $A + \varepsilon_0 \ge A : \varepsilon_0 > 0$ if it is complete in H and

$$A\|\boldsymbol{c}\|^2 \le \left\| \sum_{n\in\mathbb{Z}} c(n)\varphi_n \right\|^2 \le (A + \varepsilon_0)\|c\|^2, c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2, \varepsilon_0 > 0,$$

where $\|\boldsymbol{c}\|^2 = \sum_{n\in\mathbb{Z}} |\mathrm{c(n)}|^2$.

We let $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$ be the series of the shift invariant spaces, where $\sum_{d=1}^{m} \varphi(t_d)$ is a series of a Riesz generators, that is, $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a series of a Riesz bases of $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$. Then

$$\sum_{d=1}^{m} V\left(\varphi(t_d)\right) = \left\{\sum_{d=1}^{m} (\boldsymbol{c} * \varphi)(t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} c(n)\varphi(t_d - n) : C = \{c(n)\}_{n\in\mathbb{Z}} \in l^2\right\}.$$

It is well known see [5] that $\sum_{d=1}^{m} \varphi(t_d)$ is a series of a Riesz generators if and only if there are constant A such that $A \leq G_\varphi(\xi) \leq A + \varepsilon_0 \ \ a.e. on\ [0, 2\pi]$. In this case, $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a series of a Riesz bases of $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$ with bound $\varepsilon_0 > 0$. For any $c = \{c(n)\}_{n\in\mathbb{Z}}$ and $d = \{d(n)\}_{n\in\mathbb{Z}}$ in l^2, the discrete convolution product of c and d is defined by

$c * d = \{(c * d)(n) = \sum_{n\in\mathbb{Z}} c\,(k) d(n - k)\}$.Then $\hat{c}^*\,(\xi)\,\hat{d}^{\,*}(\xi)$ belongs to $L^1[0, 2\pi]$ and its Fourier series is $(c * d)(n)e^{-in\,\xi}$ so that

$$\int_0^{2\pi} \left|\hat{c}^*\,(\xi)\,\hat{d}^{\,*}(\xi)\right|^2 d\,\xi = 2\,\pi\,||c * d||^2. \tag{1}$$

Proposition 2.1: Let $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$and $A > 0$. Then

(a) $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a Bessel sequence with a Bessel bound $A + \varepsilon_0$ if and only if $2\,\pi\,G_\varphi\,(\xi) \leq A + \varepsilon_0$ a.e. on $[0, 2\pi]$,

(b) $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a frame sequence with frame bounds $(A, A + \varepsilon_0)$ if and only if

$$A \leq 2\,\pi\,G_\varphi\,(\xi) \leq A + \varepsilon_0\ a.e.\ on\ E_\varphi\ , \tag{2}$$

(c) $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a Riesz sequence with Riesz bounds $(A, A + \varepsilon_0)$ if and only if $A \leq 2\,\pi\,G_\varphi\,(\xi).(A + \varepsilon_0)$ a.e. on $[0, 2\,\pi]$,

(d) $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is an orthonormal sequence if and only if $2\,\pi\,G_\varphi\,(\xi) = 1$ a.e. on $[0, 2\,\pi]$.

Proof: (See [6]) For each $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$ and $c = \{c(n)\}_{n\in\mathbb{Z}} \in l^2$, let $T(c) = (c * \varphi)(t) = \sum_{k\in\mathbb{Z}}\sum_{d=1}^{m} c(k)\,\varphi\,(t_d - k)$ be the semi-discrete convolution product of c and $\sum_{d=1}^{m} \varphi(t_d)$, which may or may not converge in $L^2(\mathbb{R})$. In terms of the operator T, called the pre-frame operator of $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$, (see [6]): $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$is a Bessel sequence with a Bessel bound B if and only if T is a bounded linear operator from l^2 into $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$ and $||T(c)||^2{}_{L^2(\mathbb{R})} \leq A + \varepsilon_0||c||^2$, $c \in l^2$, $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a frame sequence with frame bounds $(A, A + \varepsilon_0)$ if and only if T is a bounded linear operator from l^2 onto $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$ and

$$A||c||^2 \leq ||T(c)||^2{}_{L^2(\mathbb{R})} \leq (A + \varepsilon_0)||c||^2, c \in N(T)^\perp , \tag{3}$$

where $N(T) = Ker\,T = \{c \in l^2 : T(c) = 0\}$ and $N(T)^\perp$ is the orthogonal complement of $N(T)$ in l^2, $\{\sum_{d=1}^{m} \varphi(t_d - n) : n \in \mathbb{Z}\}$ is a Riesz sequence with Riesz bounds $(A, A + \varepsilon_0)$ if and only if T is an isomorphism from l^2 onto $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$ and

$\boldsymbol{A}||\boldsymbol{c}||^{\boldsymbol{2}} \leq ||\boldsymbol{T}(\boldsymbol{c})||^{\boldsymbol{2}}{}_{\boldsymbol{L^2}(\mathbb{R})} \leq (A + \varepsilon_0)||c||^2, \boldsymbol{c} \in \boldsymbol{l^2}$,$\{\sum_{d=1}^{m} \varphi(t_d - n) : \boldsymbol{n} \in \mathbb{Z}\}$ is an orthonormal sequence if and only if T is a unitary operator from l^2 onto $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$.

Lemma 2.2: Let $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$. If $\{\sum_{d=1}^{m} \varphi\,(t_d - n) : n \in \mathbb{Z}\}$ is a Bessel sequence, then for any

$$c = \{c(n)\}_{n\,\in\,\mathbb{Z}} \text{ in } l^2, \widehat{c * \varphi}\,(\xi) = \hat{c}^*\,(\xi)\hat{\varphi}\,(\xi) \tag{4}$$

so that

$$\|(c * \varphi)(t)\|^2{}_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} |\hat{c}^*\,(\xi)\hat{\varphi}(\xi)|^2 d\,\xi$$

$$= \int_0^{2\pi} |\hat{c}^*(\xi)|^2\,G_\varphi\,(\xi)d\xi\,. \tag{5}$$

Proof: See [2,18]. Let $\sum_{d=1}^{m} \varphi(t_d)$ be a frame or a Riesz generator. Then T is an isomorphism from $N(T)^\perp$ onto $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$ so that

$$\sum_{d=1}^{m} V\left(\varphi(t_d)\right) = \left\{\sum_{d=1}^{m} (c * \varphi)(t_d) : c \in l^2\right\} = \left\{\sum_{d=1}^{m} (c * \varphi)(t_d) : c \in N(T)^\perp\right\},$$

where $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m}(c * \varphi)(t_d)$ is the L^2-$limit$ of $\sum_{k\in\mathbb{Z}}\sum_{d=1}^{m} c(k)\,\varphi(t_d - k)$. Applying (5), we have $N(T) = \{c \in l^2 : \hat{c}^*(\xi) = 0$ a.e. on $E_\varphi\}$ so that

$$N(T)^{\perp} = \{c \in l^2 : \hat{c}^*(\xi) = 0 \; a.e. on\; N_\varphi\}. \tag{6}$$

Proposition 2.3: putting $\sum_{d=1}^{m} \varphi(t_d) \in L^2(\mathbb{R})$ be a frame generator and $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m}(c * \varphi)(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d))$ hence $c \in l^2$. Then c $\in N(T)^{\perp}$ if and only if $c(k) = \langle f(t_d), \psi(t_d - k)\rangle_{L^2(\mathbb{R})}, k \in \mathbb{Z}, 1 \le d \le m$, hence $\{\sum_{d=1}^{m}\psi(t_d - k) : k \in \mathbb{Z}\}$ is the canonical dual frame of $\{\sum_{d=1}^{m}\varphi(t_d - k) : k \in \mathbb{Z}\}$.

Proof: Applying (4) for any $\sum_{d=1}^{m} f(t_d) = (c * \varphi)(t) \in \sum_{d=1}^{m} V(\varphi(t_d))$,

$$\sum_{d=1}^{m}\langle f(t_d), \psi(t_d - k)\rangle_{L^2(\mathbb{R})} = \langle \hat{c}^*(\xi)\hat{\varphi}(\xi), e^{-ik\xi}\,\hat{\psi}(\xi)\rangle_{L^2(\mathbb{R})}$$

$$= \langle \hat{c}^*(\xi)\,\hat{\varphi}(\xi), \frac{\hat{\varphi}(\xi)}{2\pi G_\varphi(\xi)}\chi supp\, G_\varphi(\xi) e^{-ik\xi}\rangle_{L^2(\mathbb{R})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \hat{c}^*(\xi)\,\chi_{E_\varphi}(\xi) e^{ik\xi}\, d\xi, k \in \mathbb{Z}$$

since $\hat{\psi}(\xi) = \frac{\hat{\varphi}(\xi)}{2\pi G_\varphi(\xi)}\chi supp\, G_\varphi(\xi)$ (see [13]), where $\chi_E(\xi)$ is the characteristic function of a subset E of $\mathbb{R}$. Hence

$$\sum_{d=1}^{m}\sum_{k\in\mathbb{Z}}\langle f(t_d), \psi(t_d - k)\rangle_{L^2(\mathbb{R})} e^{-ik\xi} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\left(\int_0^{2\pi}\hat{c}^*(\xi)\chi_{E_\varphi}(\xi)e^{ik\xi}\, d\xi\right)$$

$$= \hat{c}^*(\xi)\,\chi_{E_\varphi}(\xi).$$

Now, $c \in N(T)^{\perp}$ if and only if $\hat{c}^*(\xi) = 0$ a.e. on N_φ (see (6)).

That is, $\hat{c}^*(\xi) = \hat{c}^*(\xi)\,\chi_{E_\varphi}(\xi)$ a.e. on $[0, 2\pi]$. Hence the conclusion follows. A Hilbert space H consisting of complex-valued functions on a set E is called a reproducing kernel Hilbert space (RKHS in short) if the point evaluation $l_t(f) = f(t)$ is a bounded linear functional on H for each t in E. In an RKHS H, there is a function $k(s,t)$ on $E \times E$, called the reproducing kernel of H satisfying

(i) $k(\cdot, s) \in H$ for each s in E,

(ii) $\langle f(t), k(t,s)\rangle = f(s), f \in H$.

Moreover, any norm converging sequence in an RKHS H converges also uniformly on any subset of E, on which $k(t,t)$ is bounded (see [4]).

If a series of shift invariant space $\sum_{d=1}^{m} V(\varphi(t_d))$ with a frame generator $\sum_{d=1}^{m}\varphi(t_d)$ is an RKHS, then its reproducing kernel is given by

$$\sum_{d=1}^{m} k(t_d, s) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\varphi(t_d - n)\overline{\varphi(s - n)} = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\varphi(t_d - n)\,\overline{\varphi(s - n)} \tag{7}$$

where $\{\sum_{d=1}^{m}\psi(t_d - n) : n \in \mathbb{Z}\}$ is the canonical dual frame of $\{\sum_{d=1}^{m}\varphi(t_d - n) : n \in \mathbb{Z}\}$. We now find conditions on the generator $\sum_{d=1}^{m}\varphi(t_d)$ under which $\sum_{d=1}^{m} V(\varphi(t_d))$ can be recognized as an RKHS. Since all functions in an RKHS must be pointwise well defined, we only consider generators $\sum_{d=1}^{m}\varphi(t_d)$ satisfying $\sum_{d=1}^{m}\varphi(t_d)$ is a complex valued square integrable

$$\text{function well defined every where on } \mathbb{R}\,. \tag{8}$$

If $\sum_{d=1}^{m} V(\varphi(t_d))$ is recognizable as an RKHS with the reproducing kernel $\sum_{d=1}^{m} k(t_d, s)$ as in (7), where $\sum_{d=1}^{m}\varphi(t_d)$ is a frame generator satisfying (8), hence

$$\Phi(s) = \sum_{n\in\mathbb{Z}}|\varphi(s - n)|^2 = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}\left|\langle(k(t_d, s), \varphi(t_d - n)\rangle_{L^2(\mathbb{R})}\right|^2$$

$$\le (A + \varepsilon_0)||K(\cdot, s)||^2_{L^2(\mathbb{R})} = (A + \varepsilon_0)k(s,s), s \in \mathbb{R},$$

therefore $A + \varepsilon_0$ is an upper frame bound of $\{\sum_{d=1}^{m}\varphi(t_d - n) : n \in \mathbb{Z}\}$. Hence

$$\sum_{d=1}^{m}\Phi(t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}|\varphi(t_d - n)|^2 < \infty \text{ for any } t \text{ in } \mathbb{R}\,. \tag{9}$$

Conversely, we have:

III. Asymmetric multi-channel sampling Lemmas

The aim of this paper is as follows (see [11]). Let $\{L_{(1+\varepsilon_1)}\,[\cdot]: \varepsilon_1 \geq 0\}$ be N LTI (linear time-invariant) systems with impulse responses $\{\sum_{d=1}^{m} L_{(1+\varepsilon_1)}(t_d): \varepsilon_1 \geq 0\}$. Develop a stable series of shifted multi-channel sampling formula for any signal $\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d))$ using discrete sample values from $\{\sum_{d=1}^{m} L_{(1+\varepsilon_1)}(t_d): \varepsilon_1 \geq 0\}$, where each channeled signal $\sum_{d=1}^{m} L_{(1+\varepsilon_1)}\,[\,f\,](t_d)$ for $\varepsilon_1 \geq 0$ is assigned with a distinct sampling rate

$$\sum_{d=1}^{m} f(t_d) = \sum_{\varepsilon_1=0}^{N}\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} L_{(1+\varepsilon_1)}\,[f\,]\big(\sigma_{(1+\varepsilon_1)} + (1+\varepsilon_2)_{(1+\varepsilon_1)}n\big) s_{d\,(1+\varepsilon_1),n}\,(t_d),$$

$$\sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V\,(\varphi(t_d))\,, \tag{10}$$

where $\left\{\sum_{d=1}^{m} s_{d\,(1+\varepsilon_1),n}(t_d): \varepsilon_1 \geq 0, n \in \mathbb{Z}\right\}$ is a series of frames or a Riesz basis of $\sum_{d=1}^{m} V\left(\varphi(t_d)\right)$, $\{(1+\varepsilon_2)_{(1+\varepsilon_1)} : \varepsilon_1 \geq 0\}$ are positive integers, and $\{\sigma_{(1+\varepsilon_1)}: \varepsilon_1 \geq 0\}$ are real constants. Note that the series of shifting of sampling instants is unavoidable in some uniform sampling [11] and arises naturally when we allow rational sampling periods in (10). Here, we assume that each $L_{(1+\varepsilon_1)}\,[\cdot]$ is one of the following three types: the impulse response $\sum_{d=1}^{m} l(t_d)$ of an LTI system is such that

(i) $\sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m} \delta(t_d + a)\,, a \in \mathbb{R}$ or

(ii) $\sum_{d=1}^{m} l(t_d) \in L^2(\mathbb{R})$ or

(iii) $\hat{l}(\xi) \in L^\infty(\mathbb{R}) \cup L^2(\mathbb{R})$ when

$H_\varphi(\xi)=\sum_{n\in\mathbb{Z}} |\,\hat{\varphi}(\xi + 2n\pi)| \in L^2[0, 2\pi]$ .For type (i),

$\sum_{d=1}^{m} L[\,f\,](t_d) = \sum_{d=1}^{m} f(t_d + a), f \in L^2(\mathbb{R})$ so that $L[\cdot]: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isomorphism. In particular, for any $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m}(\boldsymbol{c} * \varphi)(t_d) \in \sum_{d=1}^{m} V\left(\varphi(t_d)\right)$,

$\sum_{d=1}^{m} L[\,f\,](t_d) = \sum_{d=1}^{m}(\boldsymbol{c} * \psi)(t_d)$ converges absolutely on $\mathbb{R}$ since

$$\sum_{d=1}^{m} C_\psi(t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} |\psi(t_d + n)|^2 < \infty, \sum_{d=1}^{m} t_d \in \mathbb{R}\,, \text{where}$$

$\sum_{d=1}^{m} \psi(t_d) = \sum_{d}^{m} L[\varphi](t_d) = \sum_{d=1}^{m} \varphi(t_d + a)$. For types (ii) and (iii), we have the following results (see [11]):

Lemma 3.1. Putting $L[\cdot]$ be an LTI system with the impulse response $\sum_{d=1}^{m} l(t_d)$ of the type (ii) or (iii) as above and

$\sum_{d=1}^{m} \psi(t_d) = \sum_{d=1}^{m} L[\varphi](t_d) = \sum_{d=1}^{1}(\varphi * l)(t_d)$ .Then

(a) $\displaystyle\sum_{d=1}^{m} \psi(t_d) \in C_\infty(\mathbb{R}) = \left\{\sum_{d=1}^{m} u(t_d) \in C(\mathbb{R}): \lim_{\sum_{d=1}^{m}|t_d|\to\infty} \sum_{d=1}^{m} u(t_d) = 0\right\}$,

(b) $sup_{\mathbb{R}} \sum_{d=1}^{m} C_\psi(t_d) < \infty$;

(c) for each $\sum_{d=1}^{m} f(t_d) = \sum_{d=1}^{m}(\boldsymbol{c} * \varphi)(t_d) \in \sum_{d=1}^{m} V\left(\varphi(t_d)\right)$,

$\sum_{d=1}^{m} L[\,f\,](t_d) = \sum_{d=1}^{m}(\boldsymbol{c} * \psi)(t_d)$ converges absolutely and uniformly on $\mathbb{R}$.

Hence $\sum_{d=1}^{m} L[\,f\,](t_d) \in C(\mathbb{R})$.

Proof . Suppose that $\sum_{d=1}^{m} l(t_d) \in L^2(\mathbb{R})$. Then $\sum_{d=1}^{m} \psi(t_d) \in C_\infty(\mathbb{R})$ by the Riemann–Lebesgue lemma since $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^1(\mathbb{R})$. Since

$$\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right| \leq G_\varphi(\xi)^{\frac{1}{2}} G_l(\xi)^{\frac{1}{2}} \quad ,$$

$$\left\|\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(\xi + 2n\pi)\right|\right\|^2_{L^2[0,2\pi]} \leq \int_0^{2\pi} G_\varphi(\xi)\, G_l(\xi)\,d\xi \;\; \leq 2\pi \left\|G_\varphi(\xi)\right\|_{L^\infty(\mathbb{R})} \|l\|^2{}_{L^2(\mathbb{R})}.$$

Thus for any $\sum_{d=1}^{m} t_d$ in $\mathbb{R}$, we have by the Poisson summation formula (se [1])

$$\sum_{n\in\mathbb{Z}} \hat{\psi}(\xi + 2n\pi) \prod_{d=1}^{m} e^{it_d(\xi+2n\pi)} = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} \psi(t_d + n)\, e^{-in\xi} \text{ in } L^2\;[0,2\pi]$$

Therefore any $\sum_{d=1}^{m} t_d$ in $\mathbb{R}$

$$\sum_{d=1}^{m} C_\psi(t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} |\psi(t_d + n)|^2 = \frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}}\sum_{d=1}^{m} \psi(t_d + n)\, e^{-in\xi}\right\|^2_{L^2\;[0,2\pi]}$$

$$= \frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}}\hat{\psi}(\xi + 2n\pi)\prod_{d=1}^{m}e^{it_d(\xi+2n\pi)}\right\|^2_{L^2\ [0,2\pi]}$$
$$\leq \left\|G_\varphi(\xi)\right\|_{L^\infty(\mathbb{R})}\|l\|^2_{L^2(\mathbb{R})}.$$

By Young's inequality on the convolution product, $\|L[f]\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}\|l\|_{L^2(\mathbb{R})}$ so that $L[\cdot] : L^2(\mathbb{R}) \to L^\infty(\mathbb{R})$ is a bounded linear operator. Where

$$\sum_{d=1}^{m}f(t_d) = \sum_{d=1}^{m}(c * \varphi)(t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}c(n)\varphi(t_d - n) \in \sum_{d=1}^{m}V(\varphi(t_d)),$$
$$\sum_{d=1}^{m}L[f](t_d) = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}c(n)L[\varphi(t_d - n)] = \sum_{n\in\mathbb{Z}}\sum_{d=1}^{m}c(n)\psi(t_d - n),$$

which converges absolutely and uniformly on R by (b). Now assume that $H_\varphi(\xi) \in L^2\ [0,2\pi]$. The case $\hat{l}(\xi) \in L^2(\mathbb{R})$ is reduced to type (ii). So let $\hat{l}(\xi) \in L^\infty(\mathbb{R})$. Then $\hat{\varphi}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ so that $\hat{\psi}(\xi) = \hat{\varphi}(\xi)\hat{l}(\xi) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and so $\psi(\xi) \in C_\infty(R) \cap L^2(\mathbb{R}))$. Since

$\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(\xi + 2n\pi)\right| \leq \|l\|_{L^\infty(\mathbb{R})}H_\varphi(\xi)$, we have again

by the Poisson summation formula

$$\sum_{d=1}^{m}C_\psi(t_d) = \frac{1}{2\pi}\left\|\sum_{n\in\mathbb{Z}}\hat{\psi}(\xi + 2n\pi)\prod_{d=1}^{m}e^{it_d(\xi+2n\pi)}\right\|^2_{L^2\ [0,2\pi]}$$
$$\leq \|l\|^2_{L^\infty(\mathbb{R})}\left\|H_\varphi(\xi)\right\|^2_{L^2\ [0,2\pi]}$$

so that $\sup_{\mathbb{R}}\sum_{d=1}^{m}C_\psi(t_d) < \infty$. For any $f \in L^2(\mathbb{R})$,

$$\sum_{d=1}^{m}\|L[f](t_d)\|_{L^2(\mathbb{R})} = \|f * l\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\left\|\hat{f}(\xi)\hat{l}(\xi)\right\|_{L^2(\mathbb{R})}$$
$$\leq \left\|\hat{l}\right\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

Hence $L[\cdot] : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator so that for any $\sum_{d=1}^{m}f(t_d) = \sum_{d=1}^{m}(\boldsymbol{c} * \varphi)(t_d) \in \sum_{d=1}^{m}V(\varphi(t_d))$, $\sum_{d=1}^{m}L[f](t_d) = \sum_{d=1}^{m}(\boldsymbol{c} * \psi)(t_d)$ converges in $L^2(\mathbb{R})$. By (b), $\sum_{d=1}^{m}(\boldsymbol{c} * \psi)(t_d)$ also converges absolutely and uniformly on $\mathbb{R}$.

By Lemma 3.2(b), $\sum_{d=1}^{m}\psi(t_d) \in L^2(\mathbb{R})$. However, $\sum_{d=1}^{m}(\boldsymbol{c} * \psi)(t_d)$ may not converge in $L^2(\mathbb{R})$ unless $\{\sum_{d=1}^{m}\psi(t_d - n) : n \in \mathbb{Z}\}$ is a Bessel sequence.

Lemma 3.2(b) improves Lemma 1 in [9], in which the proof uses $\sum_{d=1}^{m}l(t_d) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $sup_{\mathbb{R}}\sum_{d=1}^{m}C_\varphi(t_d) < \infty$, and the integral version of Minkowski inequality. Note that the condition $H_\varphi(\xi) \in L^2[0,2\pi]$ implies $\sum_{d=1}^{m}\varphi(t_d) \in L^2(\mathbb{R}) \cap C_\infty((\mathbb{R})$ and $sup_{\mathbb{R}}\sum_{d=1}^{m}C_\varphi(t_d) < \infty$. (see [1]). Note also that $H_\varphi(\xi) \in L^2[0,2\pi]$ if $\hat{\varphi}(\xi) = O\left((1 + |\xi|)^{-(1+\varepsilon_2)}\right), (1+\varepsilon_2)_{(1+\varepsilon_1)} > 1, \varepsilon_1 \geq 0$, which holds e.g. for $\sum_{d=1}^{m}\varphi_n(t_d) = \sum_{d=1}^{m}(\varphi_0 * \varphi_{n-1})(t_d)$ the cardinal B-spline of degree n (≥ 1), where $\varphi_0 = \sum_{d=1}^{m}\chi_{[0,1)}(t_d)$. We have as a consequence of Lemma 3.2: Let $L[\cdot]$ be an LTI system with impulse response $\sum_{d=1}^{m}l(t_d)$ of type (i) or (ii) or (iii) as above and $\sum_{d=1}^{m}\psi(t_d) = \sum_{d=1}^{m}L[\varphi](t_d)$. Then for any $\sum_{d=1}^{m}f(t_d) = \sum_{d=1}^{m}(\mathcal{J}F)(t_d) \in \sum_{d=1}^{m}V\left(\varphi(t_d)\right), F(\xi) \in L^2[0,2\pi]$

$$\sum_{d=1}^{m}L[f](t_d) = \sum_{d=1}^{m}\langle(\xi), \frac{1}{2\pi}\overline{Z_\psi(t_d,\xi)}\rangle_{L^2[0,2\pi]} \tag{11}$$

since $L[\cdot]$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ or $L^\infty(\mathbb{R})$ and $\{\sum_{d=1}^{m}\psi(t_d - n) : n \in \mathbb{Z}\} \in l^2$, $\sum_{d=1}^{m}t_d \in \mathbb{R}$. Let $\sum_{d=1}^{m}\psi_{(1+\varepsilon_1)}(t_d) = \sum_{d=1}^{m}L_{(1+\varepsilon_1)}[\varphi](t_d)$ and $g_{(1+\varepsilon_1)}(\xi) = \frac{1}{2\pi}Z_{\psi_{(1+\varepsilon_1)}}(\sigma_{(1+\varepsilon_1)},\xi)$, $\varepsilon_1 \geq 0$. Then we have by (11)

$$L_{(1+\varepsilon_1)}[f]\left(\sigma_{(1+\varepsilon_1)} + (1+\varepsilon_2)_{(1+\varepsilon_1)}n\right) = \langle F(\xi), \frac{1}{2\pi}Z_{\psi_{(1+\varepsilon_1)}}\left(\sigma_{(1+\varepsilon_1)} + (1+\varepsilon_2)_{(1+\varepsilon_1)}n, \xi\right)\rangle_{L^2\ [0,2\pi]}$$

$$= \langle F(\xi), \overline{g_{(1+\varepsilon_1)}(\xi)}e^{-i(1+\varepsilon_2)_{(1+\varepsilon_1)}n\xi}\rangle_{L^2\ [0,2\pi]} \tag{12}$$

for any $\sum_{d=1}^{m}f(t_d) = \sum_{d=1}^{m}(\mathcal{J}F)(t_d) \in \sum_{d=1}^{m}V(\varphi(t_d))$ and $\varepsilon_1 \geq 0$. Then by (12) and the isomorphism $\mathcal{J}$ from $L^2\ [0,2\pi]$ onto $\sum_{d=1}^{m}V(\varphi(t_d))$, the sampling expansion (10) is equivalent to

$$F(\xi) = \sum_{\varepsilon_1=0}^{N} \sum_{n\in\mathbb{Z}} \langle F(\xi), \overline{g_{(1+\varepsilon_1)}(\xi)} e^{-i(1+\varepsilon_2)_{(1+\varepsilon_1)} n\xi} \rangle_{L^2[0,2\pi]} S_{(1+\varepsilon_1),n}(\xi),$$

$F(\xi) \in L^2[0,2\pi]$, where $\{S_{(1+\varepsilon_1),n}(\xi): \varepsilon_1 \geq 0, n \in \mathbb{Z}\}$ is a series of frames or a Riesz basis of $L^2[0,2\pi]$. This observation leads us to consider the problem when is $\{\overline{g_{(1+\varepsilon_1)}(\xi)} e^{-i(1+\varepsilon_2)_{(1+\varepsilon_1)} n\xi}: \varepsilon_1 \geq 0, n \in \mathbb{Z}\}$ a series of frames or a Riesz basis of $L^2[0,2\pi]$. Note that

$\{\overline{g_{(1+\varepsilon_1)}(\xi)} e^{-i(1+\varepsilon_2)_{(1+\varepsilon_1)} n\xi}: \varepsilon_1 \geq 0, n \in \mathbb{Z}\} =$

$$\left\{\overline{g_{(1+\varepsilon_1),m_{(1+\varepsilon_1)}}(\xi)} e^{-i(1+\varepsilon_2)n\xi}: \varepsilon_1 \geq 0, 1 \leq m_{(1+\varepsilon_1)} \leq \frac{(1+\varepsilon_2)}{(1+\varepsilon_2)_{(1+\varepsilon_1)}}, n \in \mathbb{Z}\right\}$$

where $(1+\varepsilon_2) = lcm\{(1+\varepsilon_2)_{(1+\varepsilon_1)} : \varepsilon_1 \geq 0\}$ and $g_{(1+\varepsilon_1),m_{(1+\varepsilon_1)}}(\xi) = g_{(1+\varepsilon_1)}(\xi) e^{i(1+\varepsilon_2)_{(1+\varepsilon_1)}(m_{(1+\varepsilon_1)}-1)\xi}$ for $\varepsilon_1 \geq 0$. Let D be the unitary operator from $L^2[0,2\pi]$onto $L^2(I)^{(1+\varepsilon_2)}$, where $I = [0, \frac{2\pi}{(1+\varepsilon_2)}]$, defined by

$DF = \left[F\left(\xi + (k-1)\frac{2\pi}{(1+\varepsilon_2)}\right)\right]_{k=1}^{(1+\varepsilon_2)}$, $F(\xi) \in L^2[0,2\pi]$. We also let

$$G(\xi) = \left[Dg_{1,1}(\xi), \ldots, Dg_{1,\frac{(1+\varepsilon_2)}{(1+\varepsilon_2)_1}}(\xi), \ldots, Dg_{N,1}(\xi), \ldots, Dg_{N,\frac{(1+\varepsilon_2)}{(1+\varepsilon_2)_N}}(\xi)\right]^T \quad (13)$$

be a $\left(\sum_{\varepsilon_1=0}^{N} \frac{(1+\varepsilon_2)}{(1+\varepsilon_2)_{(1+\varepsilon_1)}}\right) \times (1+\varepsilon_2)$ matrix on I and $\lambda_m(\xi), \lambda_M(\xi)$ be the smallest and the largest eigenvalues of the positive semi-definite $(1+\varepsilon_2) \times (1+\varepsilon_2)$ matrix $G(\xi)^* G(\xi)$, respectively.

Lemma 3.2 : Let $F(\xi) \in L^1(\mathbb{R})$ so that $f(t) = \mathcal{F}^{-1}[F](t) \in C(\mathbb{R})$ and $0 \leq \sigma < 1$. Then $\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)$ converges absolutely in $L^1[0,2\pi]$ and

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi) \sim \frac{1}{\sqrt{2\pi}} Z_f(\sigma,\xi)$$
$$= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} f(\sigma + n) e^{-in\xi} \quad (14)$$

which means that $\frac{1}{\sqrt{2\pi}} Z_f(\sigma,\xi)$ is the Fourier series expansion of $\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)$. If moreover $\sum_{n\in Z} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)$ converges in $L^2[0,2\pi]$ or equivalently $\{f(\sigma + n)\}_{n\in\mathbb{Z}} \in l^2$, then

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi) = \frac{1}{\sqrt{2\pi}} Z_f(\sigma,\xi) \text{ in } L^2[0,2\pi]. \quad (15)$$

Proof: Assume that $(\xi) \in L^1(\mathbb{R})$. Then

$$\sum_{n\in\mathbb{Z}} \left\| e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi) \right\|_{L^1[0,2\pi]} = \sum_{n\in\mathbb{Z}} \int_0^{2\pi} |F(\xi + 2n\pi)| \, d\xi$$
$$= \sum_{n\in\mathbb{Z}} \int_{2n\pi}^{2(n+1)\pi} |F(\xi)| d\xi = \int_{-\infty}^{+\infty} |F(\xi)| d\xi$$

so that

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi) \text{ converges absolutely in } L^1[0,2\pi] .$$

Hence

$$\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)$$
$$\sim \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \langle \sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi), e^{-ik\xi} \rangle_{L^2[0,2\pi]} \, e^{-ik\xi},$$

where

$$\langle \sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi), e^{-ik\xi} \rangle_{L^2[0,2\pi]}$$

$$= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)\ e^{ik\xi}\ d\xi$$

$$= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)\ e^{ik\xi}\ d\xi$$

$$= \int_{-\infty}^{+\infty} F(\xi)\ e^{i(\sigma+k)\xi}\ d\xi = \sqrt{2\pi}\, f(\sigma + k)$$

by the Lebesgue dominated convergence theorem. Hence (14) holds. Now assume that $F(\xi) \in L^1(\mathbb{R})$ and $\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)$ converges in $L^2[0,2\pi]$. Then (15) becomes an orthonormal basis expansion of $\sum_{n\in\mathbb{Z}} e^{i\sigma(\xi+2n\pi)} F(\xi + 2n\pi)$ in $L^2[0,2\pi]$ so that (15) holds.

Corollary 3.3: (see [3]). If $F(\xi)$ is measurable on $\mathbb{R}$ and $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges absolutely in $L^2[0,2\pi]$, then

$$\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi) = \frac{1}{\sqrt{2\pi}} Z_f(0,\xi) \quad \text{where} \quad f(t) = \mathcal{F}^{-1}[F](t).$$

Proof：Assume that $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges absolutely in $L^2[0,2\pi]$. Then $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges absolutely also in $L^1[0,2\pi]$ so that $F(\xi) \in L^1[0,2\pi]$ and $\sum_{n\in\mathbb{Z}} F(\xi + 2n\pi)$ converges in $L^2[0,2\pi]$. Hence the conclusion follows from Lemma 3.1 for $\sigma = 0$.

Example 3.4: (see [1],[19] and [15]). Let $\sum_{d=1}^{m} \varphi_0(t_d) = \sum_{d=1}^{m} \chi_{[0,1)}(t_d)$ and

$$\sum_{d=1}^{m} \varphi_n(t_d) = \sum_{d=1}^{m} \varphi_{n-1}(t_d) * \varphi_0(t_d) = \int_0^1 \sum_{d=1}^{m} \varphi_{n-1}(t_d - s)ds, n \ge 1, \sum_{d=1}^{m}(\varphi_n(t_d) = \sum_{d=1}^{m} B_{n+1}(t_d))$$

be the cardinal B-spline of degree n. Then

$\widehat{\varphi_n}(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{-i\xi}}{i\xi}\right)^{n+1}$ and $|\hat{\varphi}n(\xi)| = \frac{1}{\sqrt{2\pi}}\left|sinc\ \frac{\xi}{2\pi}\right|^{n+1}, n \ge 0.$

It is known in [5] that $\sum_{d=1}^{m} \varphi_0(t_d)$ are an orthonormals generators and $\sum_{d=1}^{m}(\varphi_n(t_d)$ for $n \ge 1$is a continuous Riesz generator. Moreover since $\sum_{d=1}^{m}(\varphi_n(t_d)$ has compact support,

$$sup_{\mathbb{R}} \sum_{d=1}^{m} \Phi_n(t_d) = sup_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi_n(t_d - k)|^2 < \infty \text{ so that } \sum_{d=1}^{m} V(\varphi(t_d)) \text{ is an RKHS for}$$

$n \ge 0$. Since $\varphi_0(\sigma + n) = \delta_{0,n}$ for $n \in \mathbb{Z}$ and $0 \le \sigma < 1$, $Z_{\varphi_0}(\sigma,\xi) = 1$ so that by Theorem 3.3 in [1], we have an orthonormal expansion

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} f(\sigma + n)\varphi_0(t_d - n) \quad , \quad f \in \sum_{d=1}^{m} V(\varphi_0(t_d))$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$ since

$$\sum_{d=1}^{m} \Phi_0(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} |\varphi_0(t_d - n)|^2 = 1 \quad \text{on } \mathbb{R}.$$

For $\sum_{d=1}^{m} \varphi_1(t_d) = t\chi_{[0,1)}(t_d) + (2 - t)\sum_{d=1}^{m} \chi_{[1,2)}(t_d)$, and $0 \le \sigma < 1$, $\varphi_1(t) = \sigma$, $\varphi_1(\sigma + 1) = 1 - \sigma, \varphi_1(\sigma + n) = 0$ for $n \ne 0,1$ so that $Z_{\varphi_1}(\sigma,\xi) = \sigma + (1-\sigma)e^{-i\xi}$. Then $\left\|Z_{\varphi_1}(\sigma,\xi)\right\|_0 = |2\sigma - 1|$ and $\left\|Z_{\varphi_1}(\sigma,\xi)\right\|_\infty$=1. Hence by Theorem 3.3 in [1], for any σ with

$0 \leq \sigma < 1$ and $\sigma \neq \frac{1}{2}$,

we have a Riesz basis expansion

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} f(\sigma + n)S(t_d - n) \quad , \qquad f \in \sum_{d=1}^{m} V\,(\varphi_1\,(t_d))$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$. For

$\sum_{d=1}^{m} \varphi_2(t_d) = \frac{1}{2}t^2 \sum_{d=1}^{m} \chi_{[1,2)}\,(t_d) + \frac{1}{t^2 2}(6t - 2 - 3)\sum_{d=1}^{m} \chi_{[1,2)}\,(t_d) + \frac{1}{2}(3-t)^2 \sum_{d=1}^{m} \chi_{[1,2)}\,(t_d)$, it is known (see [1] and [11]) that

$\left\|Z_{\varphi_2}(0,\xi)\right\|_0 = 0$ but $\frac{1}{2} \leq \left\|Z_{\varphi_2}\left(\frac{1}{2}\ ,\xi\right)\right\|_0 < \left\|Z_{\varphi_2}\left(\frac{1}{2}\ ,\xi\right)\right\|_\infty \leq 1$ so that there is a Riesz basis expansion

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} f\left(\frac{1}{2} + n\right)S(t_d - n)\,, f \in \sum_{d=1}^{m} V\,(\varphi_2\,(t_d)) \qquad (16)$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$. Since the optimal upper Riesz bound of the Riesz sequence $\{\varphi_2(t_d - k) :\, k, d \in \mathbb{Z}\}$ is 1 (see [5]), we have for the sampling series (16)

$$\sum_{d=1}^{m} \|E_n(f)(t_d)\|^2{}_{L^2(\mathbb{R})} \;\leq\; 4\sum_{|k|>n} \left|f\left(\frac{1}{2}+k\right)\right|^2 , f \in \sum_{d=1}^{m} V\,(\varphi_2\,(t_d))\ .$$

On the other hand, we have

$$H\varphi_2\,(\xi) = \sum_{k\in\mathbb{Z}} |\,\hat{\varphi}_2\,(\xi + 2k\pi)| = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left|sinc\left(\frac{\xi}{2\pi}+k\right)\right|^3$$

$$\leq \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \left|sinc\left(\frac{\xi}{2\pi}+k\right)\right|^2 = \frac{1}{\sqrt{2\pi}}.$$

Example 3.5: (See [1]) Let $\sum_{d=1}^{m} \varphi(t_d) = \prod_{d=1}^{m} e^{\frac{-t_d^2}{2}}$ be the Gauss kernel . Then $\hat{\varphi}\,(\xi) = e^{\frac{-\xi^2}{2}}$ and $0 < \|G_\varphi(\xi)\|_0 < \|G_\varphi(\xi)\|_\infty < \infty$ so that $\sum_{d=1}^{m} \varphi(t_d)$ is a continuous Riesz generator satisfying

$$sup_{\mathbb{R}} \sum_{d=1}^{m} \Phi(t_d) = sup_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{d=1}^{m} |\,\varphi\,(t_d - k)|^2 < \infty.\text{Since } \hat{\varphi}\,(\xi) \in L^1(\mathbb{R})$$

and $\{\,\varphi\,(n)\}_{n\in\mathbb{Z}} \in l^1$, we have by Lemma 3.1

$$Z_{\,\varphi}(0,\xi) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} e^{\frac{-1}{2}(\xi + 2n\pi)^2} \text{ so that } 0 < \|Z_\varphi(\xi)\|_0 < \|Z_\varphi(\xi)\|_\infty < \infty.$$

Hence by Theorem 3.3 in [1] , $\sum_{d=1}^{m} V\,(\varphi(t_d))$ is an RKHS and there is a Riesz basis expansion

$$\sum_{d=1}^{m} f(t_d) = \sum\nolimits_{n\in\mathbb{Z}} \sum_{d=1}^{m} f(n)S(t_d - n)\,, f \in \sum_{d=1}^{m} V\,(\varphi(t_d))$$

which converges in $L^2(\mathbb{R})$ and uniformly on $\mathbb{R}$.

Corollary 3.6. (Cf. Theorem 3.2 in [19].) Let $N = 1$. Then there is a series of Riesz bases $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ such that

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\,\in\,\mathbb{Z}} \sum_{d=1}^{m} L[f](\sigma + (1+\varepsilon_2)n)s_n(t_d)\,, \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)) \quad (17)$$

if and only if $\varepsilon_2 = 0$ and

$$0 < \left\|Z_\psi(\sigma,\xi)\right\|_0 \leq \left\|Z_\psi(\sigma,\xi)\right\|_\infty\ . \qquad (18)$$

In this case, we also have

(i) $\sum_{d=1}^{m} s_n(t_d) = \sum_{d=1}^{m} s(t_d - n)\,, n \in \mathbb{Z}$,

(ii) $\hat{s}(\xi\,) = \frac{\hat{\varphi}(\xi)}{Z_\psi(\sigma,\xi)}$,

(iii) $L[s](\sigma + n) = \delta_{n,0}\,, n \in \mathbb{Z}.$ (19)

Proof . Note that for $\varepsilon_2 = 0, G(\xi) = \frac{1}{2\pi} Z_\psi(\sigma,\xi)$ and $\lambda_m(\xi) = \lambda_M(\xi) = \left(\frac{1}{2\pi}\right)^2 \left|Z_\psi(\sigma,\xi)\right|^2$ so that $0 < \alpha_G \leq \beta_G < \infty$ if and only if (18) holds. Therefore, everything except (19) follows from Theorem 3.4 in [1]. Finally applying (17) to $\sum_{d=1}^{m} \varphi(t_d)$ gives $\sum_{d=1}^{m} \varphi(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} \psi(\sigma + n)s(t_d - n)$ from which we have (19) by taking the Fourier transform. When $\sum_{d=1}^{m} l(t_d) = \sum_{d=1}^{m} \delta(t_d)$ so that $L[\cdot]$ is the identity operator, Corollary 3.6 reduces to a series of regular shifted sampling on $\sum_{d=1}^{m} V(\varphi(t_d))$ (see Theorem 3.3 in [17]).

Corollary 3. 7. Suppose $Z_\psi(2-\varepsilon_0,\xi) \in L^\infty[0,2\pi], 0 \leq \varepsilon_1 \leq q-1$,then the following are all equivalent.

(i) There is a series of frames $\{\sum_{d=1}^{m} s_n(t_d) : n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{d=1}^{m} L[f](2-\varepsilon_0)s_n(t_d), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(ii) There is a series of frames $\{\sum_{d=1}^{m} s_{(1+\varepsilon_1)}(t_d - n) : \varepsilon_1 > 0, n \in \mathbb{Z}\}$ of $\sum_{d=1}^{m} V(\varphi(t_d))$ for which

$$\sum_{d=1}^{m} f(t_d) = \sum_{n\in\mathbb{Z}} \sum_{\varepsilon_1\geq 0} \sum_{d=1}^{m} L[f](n-\varepsilon_0)s_{(1+\varepsilon_1)}(t_d - n), \sum_{d=1}^{m} f(t_d) \in \sum_{d=1}^{m} V(\varphi(t_d)).$$

(iii) $\left\| \sum_{\varepsilon_1\geq 0} |Z_\psi(2-\varepsilon_0,\xi)| \right\|_0 > 0.$

Proof: Since

$\{L[f](2-\varepsilon_0)\} = \{L[f](n-\varepsilon_0) : n \in \mathbb{Z}\}$. Now we have $\{L_{(1+\varepsilon_1)}[\cdot] : \varepsilon_1 > 0\}$ with $L_{(1+\varepsilon_1)}[\cdot] = L[\cdot], \varepsilon_1 > 0$. Then $g_{(1+\varepsilon_1)}(\xi) = \frac{1}{2\pi} Z_\psi(2-\varepsilon_0,\xi), \varepsilon_1 > 0$ and $G(\xi)^* G(\xi) = \frac{1}{(2\pi)^2} \sum_{\varepsilon_1\geq 0} \left|Z_\psi(2-\varepsilon_0,\xi)\right|^2$. There for $\alpha_G > 0$ if and only if

$$\left\| \sum_{\varepsilon_1\geq 0} |Z_\psi(2-\varepsilon_0,\xi)| \right\|_0 > 0 .$$

References

[1]. Adam zakria , Ahmed Abdallatif , Yousif Abdeltuif [17] sampling expansion in a series of shift invariant spaces ISSN 2321 3361 © 2016 IJESC

[2]. O. Christensen, An Introduction to Frames and Riesz Bases (Birkh¨auser, 2003).

[3]. J. R. Higgins, Sampling Theory in Fourier and Signal Analysis: Foundations (Oxford University Press, 1996).

[4]. G. G.Walter, A sampling theorem for wavelet subspaces, IEEE Trans. Inform. Theory 38 (1992) 881–884.

[5]. O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2001.

[6]. I. Djokovic, P.P. Vaidyanathan, Generalized sampling theorems in multiresolution subspaces, IEEE

[7]. A.G. Garcia, M.A. HernԼndez-Medina, G. Pérez-Villalón, Oversampling and reconstruction functions with compact support, J. Comput. Appl. Math. 227(2009) 245–253.

[8]. A.G. Garcia, G. Pérez-Villalón, Dual frames in L^2 (0,1) connected with generalized sampling in shift-invariant spaces, Appl. Comput. Harmon. Anal. 20(2006) 422–433.

[9]. A.G. Garcia, G. Pérez-Villalón, A. Portal, Riesz bases in L^2 (0,1) related to sampling in shift-invariant spaces, J. Math. Anal. Appl. 308 (2005) 703–713.

[10]. Y.M. Hong, J.M. Kim, K.H. Kwon, E.H. Lee, Channeled sampling in shift invariant spaces, Int. J. Wavelets Multiresolut. Inf. Process. 5 (2007) 753–767.

[11]. A.J.E.M. Janssen, The Zak transform and sampling theorems for wavelet subspaces, IEEE Trans. Signal Process. 41 (1993) 3360–3364.

[12]. S. Kang , J.M. Kim, K.H. Kwon Asymmetric multi-channel sampling in shift invariant spaces J. Math. Anal. Appl. 367 (2010) 20–28

[13]. X. Zhou and W. Sun, On the sampling theorem for wavelet subspaces, J. Fourier Anal. Appl. 5 (1999) 347–354.

[14]. A. Papoulis, Generalized sampling expansion, IEEE Trans. Circuits Syst. 24 (1977) 652–654.

[15]. L.J. Fogel, A note on the sampling theorem, IRE Trans. Inform.Theory IT-1(1995) 47–48.

[16]. C.E. Shannon, Communication in the presence of noise, Proc. Inst. Radio Eng. 37 (1949) 10–21.

[17]. J.M. Kim, K.H. Kwon, Sampling expansion in shift invariant spaces, Int. J. Wavelets Multiresolut. Inf. Process. 6 (2008) 223–248.

[18]. Y. M. Hong, J. M. Kim, K. H. Kwon and E. H. Lee, Channeled sampling in shift invariant spaces, Int. J. Wavelets Multiresolut. Inf. Process. 5 (2007) 753–767.