Implementation of Wavelet Based Transform for numerical solutions of Partial differential equations

Ali Al-Fayadh[1] and Nawar Hazim[2]

Department of Mathematics and Computer Applications, Al-Nahrain University, College of Science, Iraq
1) aalfayadh@yahoo.com 2) nawarbmw73@yahoo.com

Abstract: *Nonlinear phenomena play a major role in applied mathematics and engineering. The Burger-Fisher equation which is a mixed hyperbolic-parabolic non-linear partial differential equation occurs in various areas of applied sciences and physical applications, such as modeling of gas dynamics and fluid mechanics. In this paper, Haar wavelet method was implemented efficiently in finding the numerical solution of Burger-Fisher equation. This method shows rather rapid convergence than other existing methods. Illustrative examples suggest that using wavelet based method providing a powerful approach to find numerical solutions of Burger-Fisher equation. The comparison of numerical results with the exact solutions, and the solutions obtained using some traditional methods such as variational iteration method (VIM) show that the suggested scheme is fairly accurate and viable for solving such problems.*
Keywords: *Burger's-Fisher equation, Haar wavelet method, Variational iteration method.*

I. Introduction

Nonlinear partial differential equations (NPDEs) were resulted by mathematical modeling of many physical systems and appeared in various fields of engineering and science. The numerical solutions of NPDEs, such as the Burgers-Fisher equation, have attracted much attention due to their potential applications in various fields of fluid dynamics, heat conduction, gas dynamic, traffic flow, applied mathematics and some other fields of science [1-5]. The Burgers-Fisher equation which describes the interaction between the reaction mechanism, convection effect, and diffusion transport [6] is considered in this paper. Many numerical schemes have been proposed for obtaining approximate solutions of the Burger-fisher equation [7-12].

The wavelet series which is shown to be an efficient and accurate method to obtain the approximated solutions of the partial differential equations (PDEs) has been used over the past two decades. This method makes use of two parameters which are the dilation and the translation. The dilation parameter is used for compression of wavelet series solution and the translation parameter is used for the location of wavelet series solution in $L^2(R)$ space. The method is capable of giving approximated solution to the PDEs in both of the regular and irregular domain. Furthermore, the wavelet function is symmetrical, detects the discontinuities in the equation, and yields solution which is powerful against noise. In addition, the wavelet series solution of a PDE converges faster than the conventional methods of approximation on the basis that the wavelet function has the compact support property. On the other hand, due to the multiresolution analysis property in wavelet domain, it may obtain both time and frequency information which is not feasible with other methods of approximating PDEs.

In this work, the goal is to obtain the approximate solution of the Burger-Fisher equation by using the Haar wavelet method [13-14] and to compare the obtaining results with the exact solution and the iterative methods [15].

Let us consider the Burger-Fisher equation [10] is as follows:-

$$u_t - u_{xx} + \alpha u^\gamma u_x + \beta u(u^\gamma - 1) = 0, \qquad 0 \leq x \leq 1, \qquad t \geq 0 \ \dots\dots\dots\dots\dots (1)$$

with the initial condition given by:

$$u(x, 0) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\alpha\gamma}{2(\gamma + 1)}x\right)\right)^{\frac{1}{\gamma}} \dots\dots\dots\dots\dots\dots (2)$$

And the boundary conditions:

$$\left. \begin{array}{l} u(0, t) = \left(\frac{1}{2} + \frac{1}{2}tanh\left(\frac{\alpha\gamma}{2(\gamma + 1)}\left(\frac{\alpha}{\gamma + 1} + \frac{\beta(\gamma + 1)}{\alpha}\right)t\right)\right)^{\frac{1}{\gamma}}, \qquad t \geq 0 \\[4mm] u(1, t) = \left(\frac{1}{2} + \frac{1}{2}tanh\left[\frac{-\alpha\gamma}{2(\gamma + 1)}\left(1 - \left(\frac{\alpha}{\gamma + 1} + \frac{\beta(\gamma + 1)}{\alpha}\right)t\right)\right]\right)^{\frac{1}{\gamma}}, \ \ t \geq 0 \end{array} \right\} \dots\dots\dots (3)$$

Where $\alpha, \beta \geq 0$ and $\gamma > 0$ are given constants. If $\gamma = 1$, Eq.(1) is called the Burger-Fisher equation. When $\alpha = 0, \gamma = 1$, Eq.(1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibre and

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wall motion in liquid crystals [16]. Generalized Burger equation will be obtained when β =0. This equation when β = 0, has been used to investigate sound waves in a viscous medium by Lighthill [17]. However, it was originally introduced by Burgers [18] to model one-dimensional turbulence.

The boundary conditions (3) are taken from the exact solution which is given by [8-10]

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha\gamma}{2(\gamma+1)}\left(x - \left(\frac{\alpha}{\gamma+1} + \frac{\beta(\gamma+1)}{\alpha}\right)t\right)\right)\right)^{\frac{1}{\gamma}} \dots\dots\dots\dots\dots (4)$$

II.   Haar wavelet method

The wavelet transform or wavelet analysis is a recently developed mathematical tool for many problems. One of the popular families of wavelet is Haar wavelets [19]. In fact the Haar wavelet is the Daubechies wavelet of order 1. Due to its simplicity, the Haar wavelet had become an effective tool for solving many problems arising in many branches of sciences. It has been used since 1910 where the Hungarian mathematician Alfred Haar discovered it. The Haar wavelet is an orthonormal wavelet with compact support. Haar showed that certain square wave function could be translated and scaled to create a basis set that span $L^2(R)$. If we choose scaling function to have compact support over $0 \le x \le 1$, that is the Haar wavelet family for $x \in [0,1]$ is defined as

$$h_i(x) = \begin{cases} 1 & x \in [\xi_1, \xi_2) \\ -1 & x \in [\xi_2, \xi_3] \\ 0 & \text{otherwise} \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots (5)$$

Where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k+0.5}{m}, \quad \xi_3 = \frac{k+1}{m} \dots\dots\dots\dots\dots\dots\dots\dots (6)$$

where $m = 2^j$, $j = 0, 1, \dots, J$ indicates the level of the wavelet and the integer $k = 0, 1, \dots, m-1$ is the translation parameter. The maximal level of resolution is J. The index i in Eq. (5) is calculated from the formula $i = m+k+1$. In the case of minimal values $m = 1$, $k = 0$, we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$. The function $h_1(x)$ is the scaling function for the family of the Haar wavelets which is defined as.

$$h_1(x) = \begin{cases} 1 & x \in [\xi_1, \xi_2) \\ 0 & \text{otherwise} \end{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

The following integrals are useful for solving second order partial differential equations.

$$p_i(x) = \int_0^x h_i(x)\,dx = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2) \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3) \\ 0 & \text{otherwise} \end{cases} \dots\dots\dots\dots\dots (8)$$

$$q_i(x) = \int_0^x p_i(x)\,dx = \begin{cases} \frac{1}{2}(x - \xi_1)^2 & \text{for } x \in [\xi_1, \xi_2) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3) \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3, 1] \\ 0 & \text{otherwise} \end{cases} \dots\dots (9)$$

Any function y(x) which is square inegrable on [0,1] can be expressed in terms of Haar wavelet as follows
$$y(x) = \sum_{i=1}^{\infty} a(i)h_i(x). \qquad\qquad\dots\dots\dots\dots\dots(10)$$
Here, the expansion y(x) is an infinite series. If y(x) is approximated as pricewise constant in each sub-area, then it will be terminated at finite terms, that is
$$y(x) = \sum_{i=1}^{2M} a(i)h_i(x). \qquad\qquad\dots\dots\dots\dots\dots(11)$$
Where the wavelet coefficients a(i), i=1,2,……,2M are to be determined.

Now, consider an initial boundary value problem (IBVP) for the nonhomogeneous Burgers-Fisher equation (1) with the initial and boundary conditions
$$u(x,0) = f(x), 0 \le x \le 1, \qquad\dots\dots\dots\dots(12)$$
$$u(0,t) = g_1(t), t \ge 0, \qquad\dots\dots\dots\dots(13)$$
$$u(1,t) = g_2(t), t \ge 0, \qquad\dots\dots\dots(14)$$
Let us divided the interval [0,T] into N equal parts of length $\Delta t = \frac{T}{N}$ and denote t_s=(s-1) Δt,

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s = 1,2,…,N . We assume a Haar wavelet solution for equation (1) in the form [20],

$$\dot{u}''(x,t) = \sum_{i=1}^{2M} a_s(i) h_i(x), \qquad \text{……………(15)}$$

here the dot and prime denote differentiation with respect to t and x respectively, the row vector a_s is constant in the subinterval $t \in [t_s, t_{s+1}]$. Integrating equation (13) with respect to $\boldsymbol{t}$ in the limits $[t_s, t]$ then by integrating the resultant equation with respect to x in the limits $[0, x]$, and the resultant equation is again integrating with respect to $\boldsymbol{x}$ in the limits $[0, x]$ and differentiating the resultant equation with respect to $\boldsymbol{t}$, we obtain the following equations respectively.to $\boldsymbol{t}$, we obtain the following equations respectively.

$$u''(x,t) = (t - t_s) \sum_{i=1}^{2M} a_s(i) h_i(x) + u''(x, t_s), \qquad \text{…………..(16)}$$

$$u'(x,t) = (t, t_s) \sum_{i=1}^{2M} a_s(i) p_i(x) + u'(x, t_s) - u'(0, t_s) + u'(0, t), \qquad \text{……….(17)}$$

$$u(x,t) = (t, t_s) \sum_{i=1}^{2M} a_s(i) q_i(x) + u(x, t_s) - u(0, t) + x[u'(0, t) - u'(0, t)] + u(0, t), \quad \text{….(18)}$$

$$\dot{u}(x,t) = \sum_{i=1}^{2M} a_s(i) q_i(x) + \dot{u}(0, t) + x \dot{u}'(0, t). \qquad \text{…………(19)}$$

Using the boundary conditions(10)-(11), we have

$$u(0, ts) = g_1(t_s), \quad u(1, t_s) = g_2(t_s) \qquad \text{…………(20)}$$

$$\dot{u}(0, t) = \dot{g}_1(t), \quad \dot{u}(1, t) = \dot{g}_2(t). \qquad \text{………..(21)}$$

Putting x = 1 in equations (16) and (17) and using the conditions in (18)-(19), we obtain

$$u'(0, t) - u'(0, t_s) = -(t - t_s) \sum_{i=1}^{2M} a_s(i) q_i(1) + g_2(t) - g_1(t) - g_2(t_s) + g_1(t_s)$$
$$\text{…………(22)}$$

$$\dot{u}'(0, t) = -\sum_{i=1}^{2M} a_s(i) q_i(1) + \dot{g}_2(t) - \dot{g}_1(t) \qquad \text{………..(23)}$$

The wavelet collocation points are defined as

$$x_I = \frac{I - 0.5}{2M}, I = 1, 2, \dots, 2M. \qquad \text{………..(24)}$$

Substituting equation (18)-(21) in equations (14)-(17) and taking $x \to x_I$, $t \to t_{s+1}$, we get

$$u''(x_I, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i) h_i(x) + u''(x_I, t_s), \qquad \text{………(25)}$$

$$u'(x_I, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i) [p_i(x_I) - q_i(1)] + u'(x_I, t_s) + g_2(t_{s+1}) - g_1(t_{s+1}) - g_2(t_s) + g_1(t_s)$$
$$\text{………………………..(26)}$$

$$u(x_I, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i) [q_i(x_I) - x_I q_i(1)] + u(x_I, t_s) + x_I [g_2(t_{s+1}) - g_2(t_s)]$$
$$+ (1 - x_I)[g_1(t_{s+1}) - g_1(t_s)], \qquad \text{…………(27)}$$

$$\dot{u}(x_I, t_{s+1}) = \sum_{i=1}^{2M} a_s(i) [q_i(x_I) - x_I q_i(1)] + \dot{g}_1(t_{s+1}) + x_I [\dot{g}_2(t_{s+1}) - \dot{g}_1(t_{s+1})]. \quad \text{…(28)}$$

Using the quasilinearization technique [21] to handle the nonlinearity in equation(1).
Now, we have the following scheme

$$\sum_{i=1}^{2M} a_s(i) [q_i(x_I) - x_I q_i(1)] + \dot{g}_1(t_{s+1}) + x_I [\dot{g}_2(t_{s+1}) - \dot{g}_1(t_{s+1})] - u''(x_I, t_{s+1}) u(x_I, t_{s+1}) u'(x_I, t_s) + u'(x_I, t_{s+1}) u(x_I, t_s) = u(x_I, t_{s+1})[1 - u(x_I, t_{s+1})]$$
$$+ u(x_I, t_s) u'(x_I, t_s) \text{………..(29)}$$

This leads us from the time layer t_s to t_{s+1}

The wavelet coefficients $a_s(i)$, $i = 1, 2, \dots, 2M$ can be successively calculated from equation (29). This process is started with the initial condition (12). These coefficients are then substituted in equations (25)-(27) to obtain the approximate solutions at different time levels.

III. Numerical results

In this section, two examples are considered to check the efficiency and accuracy of the Haar wavelet method. The entire computational work has been done with the help of MATLAB software.

Example 1: Let us consider the Burger-Fisher equation (1), with the initial and boundary conditions in quations (2) and (3) respectively and the exact solution given in equation (4).

Table (1) shows the approximate solution using the proposed method against the exact solution when $\alpha = 0.01, \beta = 0.01$, and $\gamma = 1$, whereas Table (2) shows a comparison between the absolute error of the solution by the proposed method and the absolute error by using the variational iteration method [15] for different values of x and t when $\alpha = 0.001, \beta = 0.001$, and $\gamma = 1$.

Table (1)

xi	ti	Exact solution	Haar solution	Absolute Error
0.01	0.02	0.500037624999929	0.499132084264304	0.000905540735625320
0.02		0.500025124999979	0.502892009223431	0.002866884223452
0.03		0.500012624999997	0.506445479270577	0.006432854270580
0.04		0.500000125000000	0.509811685266022	0.009811560266022
0.05		0.499987625000003	0.513007881958884	0.013020256958881
0.01	0.03	0.500062687499672	0.498439354576423	0.001623332923248
0.02		0.500050187499831	0.502378619998906	0.002328432499075
0.03		0.500037687499929	0.506101165599605	0.006063478099676
0.04		0.500025187499979	0.509627059589430	0.009601872089451
0.05		0.500012687499997	0.512974341469135	0.012961653969137
0.01	0.04	0.500087749999099	0.497075433069911	0.003012316929188
0.02		0.500075249999432	0.501141883179817	0.001066633180385
0.03		0.500062749999671	0.50498498853876	0.004921748854205
0.04		0.500050249999831	0.508623956709160	0.008573706709329
0.05		0.500037749999928	0.512078840448603	0.012041090448675
0.01	0.05	0.500112812498086	0.495074644318046	0.005038168180040
0.02		0.500100312498654	0.499212224708285	0.000888087790369
0.03		0.500087812499097	0.503122230345461	0.003034417846364
0.04		0.500075312499430	0.506825558317006	0.006750345817576
0.05		0.500062812499670	0.510341379000510	0.010278566500841
0.01	0.06	0.500137874996505	0.492492007972452	0.007645867024053
0.02		0.500125374997372	0.496642393806129	0.003482981191243
0.03		0.500112874998083	0.500564964433619	0.000452089435537
0.04		0.500100374998652	0.504280738884366	0.004180363885715
0.05		0.500087874999095	0.507808607216788	0.007720732217693
0.01	0.07	0.500162937494232	0.489399727599114	0.010763209895119
0.02		0.500150437495461	0.493504097865546	0.006646339629915
0.03		0.500137937496501	0.497383931385718	0.002754006110783
0.04		0.500125437497386	0.501059969090885	0.000934531593517
0.05		0.500112937498079	0.504550852381356	0.004437914883276
0.01	0.08	0.500187999991140	0.485882659102875	0.014305340888266
0.02		0.500175499992793	0.489883504232251	0.010291995760542
0.03		0.500162999994226	0.493666521190520	0.006496478803706
0.04		0.500150499995455	0.497251880445489	0.002898619549966
0.05		0.500137999996496	0.500657713270939	0.000519713274444

Table: 2

X	T	Exact Sol.	Haar Sol.	Haar error	VIM error
0.01	0.02	0.500015019999996	0.499132084264304	0.000882935735692	0.0025031102
	0.04	0.500035039999943	0.498697734403682	0.001337305596261	0.0025081138
	0.06	0.50005509999777	0.498312881454306	0.001742218543464	0.0025131170
	0.08	0.500075079999436	0.497977989064286	0.00209709093515	0.0025181206
0.04	0.02	0.500000020000000	0.509811685266022	0.009811665266022	0.0099961959
	0.04	0.500020039999989	0.509360739108795	0.009340699108806	0.0100011899
	0.06	0.500040059999914	0.508960366604204	0.00892030660429	0.0100061907
	0.08	0.500060079999711	0.508611049610424	0.008550969610713	0.0100111915

IV.   Conclusion

The Burger–Fisher equation has been analyzed using the Haar wavelet method, and the results of this method have been compared with the variational iteration method. The proposed method shows that it is in good agreement with the exact solution and it is a better than variational iteration method.

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