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Certain Third Order Mixed Neutral Difference Equations

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Abstract: *In this paper some criteria for the oscillation of mixed type third order neutral difference equation of the form*

$$\Delta\left(a_n\Delta\left(d_n\Delta\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)\right)\right)+q_nx^{\beta}_{n+1-\sigma_1}+p_nx^{\beta}_{n+1+\sigma_2}=0$$

where β is the ratio of odd positive integers, $\tau_1,\tau_{2,}\sigma_1$ and σ_2 are non-negative integers were discussed. Examples are inserted to illustrate the main results.

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I. Introduction

The notion of nonlinear difference equation was studied intensively by R.P.Agarwal [1] and oscillatory properties were discussed by R.P. Agarwal et.al.[2], [3], [4]. Difference equations find a lot of applications in the natural sciences, technology and population dynamics.Recently there has been a lot of interest in the study of oscillatory behaviour of solutions of nonlinear difference equations. We can see this in [5-24]. Researchers carried out their researches on the oscillatory and asymptotic behaviour of solutions of difference equations with delay and neutral delay type. In this paper, we consider the third order mixed type neutral difference equation of the form

$$\Delta\left(a_n\Delta\left(d_n\Delta\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)\right)\right)+q_nx^{\beta}_{n+1-\sigma_1}+p_nx^{\beta}_{n+1+\sigma_2}=0 \qquad (1.1)$$

and $n\in N=\{n_0,n_0+1,.....\}$, n_0 is a nonnegative integer. Here Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$.

By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ which is defined for all $n\geq n_0-\theta$ and satisfies equation (1.1) for all $n\in N$ where $\theta=\max\{\tau_1,\sigma_1\}$. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non-oscillatory. A difference equation is said to be oscillatory if all of its solutions are oscillatory. Otherwise, it is non-oscillatory.

Throughout this paper, the following conditions are assumed to hold.

(H1) $\{a_n\}$ is a positive non-decreasing sequence such that $\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty.$

(H2**)** $\{d_n\}$ is a positive non-decreasing sequence.

(H3) $\{p_n\}$ and $\{q_n\}$ are positive real sequences for $n\geq n_0$.

(H4) β is the ratio of odd positive integers, τ_1,τ_2,σ_1 and σ_2 are non-negative integers.

(H5) $\{b_n\},\{c_n\}$ are real sequences such that $0\leq b_n\leq b$ and $0\leq c_n\leq c$ with $b+c<1$.

II. Preliminary Lemmas

We need the following lemmas to prove the main results. For simplicity, we use the following notations:

$$y_n=x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}, \qquad R_n=Q_n+P_n,$$
$$Q_n=\min\{q_n,q_{n-\tau_1},q_{n+\tau_2}\}, \qquad P_n=\min\{p_n,p_{n-\tau_1},p_{n+\tau_2}\}.$$
$$\eta_n=\left(\frac{d}{4}\right)^{\beta-1}\frac{k(n-\sigma_1)^{\beta}}{2^{\beta}}R_n \text{ for some } k\in(0,1) \text{ and } d>0.$$

Lemma: 2.1

Assume $A \geq 0, B \geq 0, \beta \geq 1$ and $A, B \in R$. Then $(A+B) \leq 2^{\beta-1}(A^{\beta}+B^{\beta})$.

Lemma: 2.2

Let $\{x_n\}$ be a positive solution of equation (1.1). Then there are only two cases for $n \geq n_1 \in N$ sufficiently large:

(i) $y_n > 0, \Delta y_n > 0, \Delta(d_n \Delta y_n) > 0, \Delta(a_n(d_n \Delta y_n)) \leq 0.$

(ii) $y_n > 0, \Delta y_n < 0, \Delta(d_n \Delta y_n) > 0, \Delta(a_n(d_n \Delta y_n)) \leq 0.$

Proof.

Let $\{x_n\}$ be a positive solution of equation (1.1). Then we can find an integer $n_1 \geq n_0$ such that $x_n > 0,\ x_{n-\sigma_1} > 0,\ \ x_{n+\sigma_2} > 0,\ x_{n-\tau_1} > 0,\ x_{n+\tau_2} > 0$ for all $n \geq n_1$. Then $y_n > 0$ for $n \geq n_1$.

From (1.1), we have

$$\Delta\left(a_n \Delta\left(d_n \Delta y_n\right)\right) = -q_n x^{\beta}_{n+1-\sigma_1} - p_n x^{\beta}_{n+1+\sigma_2} < 0, \qquad (2.1)$$

for $n \geq n_1$, which implies $\Delta\left(a_n \Delta\left(d_n \Delta y_n\right)\right)$ is strictly decreasing for $n \geq n_1$.

We claim $\Delta\left(d_n \Delta y_n\right) > 0$ for $n \geq n_1$. If not, then there exists $n_2 \geq n_1$ and $M < 0$ such that

$$a_n \Delta\left(d_n \Delta y_n\right) \leq a_{n_2} \Delta\left(d_{n_2} \Delta y_{n_2}\right) \leq M,$$

for $n \geq n_2$.

Summing the last inequality from n_2 to $n-1$, we get

$$d_n \Delta y_n \leq d_{n_2} \Delta y_{n_2} + M \sum_{s=n_2}^{n-1} \frac{1}{a_s},$$

which implies $\Delta y_n \to -\infty$ as $n \to \infty$. Then there exists $n_3 \geq n_2$ such that $\Delta y_n < 0$ for $n \geq n_2$. This implies $y_n \to -\infty$ as $n \to \infty$, which is a contradiction and so $\Delta\left(d_n \Delta y_n\right) > 0$ for $n \geq n_1$.

Hence the proof is complete.

Lemma 2.3.

Let $y_n > 0, \Delta y_n > 0, \Delta^2 y_n > 0, \Delta^3 y_n \leq 0$ for all $n \geq N_1 \in N$. Then for any $k \in (0,1)$ and for some integer N_1, one has $\dfrac{y_{n+1}}{\Delta y_n} \geq \dfrac{(n-N)}{2} \geq \dfrac{kn}{2}$, for $n \geq N_1 \geq N$. (2.2)

Proof.

Since

$$\Delta y_n = \Delta y_N + \sum_{s=N}^{n-1} \Delta^2 y_n,$$

we have $\Delta y_n \geq (n-N)\Delta^2 y_n$.

Summing the last inequality, we have

$$y_n \geq y_N + (n-N)\Delta y_n - y_n + y_N$$

or

$$y_{n+1} \geq \frac{(n-N)}{2}\Delta y_n \geq \frac{kn}{2}\Delta y_n$$

for $n \geq N_1 \geq N$. Hence the proof is completed.

Lemma: 2.4

Let $\{x_n\}$ be a positive solution of equation (1.1) and the corresponding y_n satisfies Lemma 2.2(ii). If

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\sum_{s=n}^{\infty}\left(\frac{1}{a_s}\sum_{t=s}^{\infty}(q_t+p_t)\right)\right)=\infty \qquad (2.3)$$

holds, then $\lim_{n\to\infty} x_n = 0$.

Proof.

Let $\{x_n\}$ be a positive solution of equation (1.1). Since $y_n > 0$ and $\Delta y_n < 0$, then $\lim_{n\to\infty} y_n = l \ge 0$ exists. We claim $l = 0$. If not, then $l > 0$.

Then for any $\in > 0$, we have $l+\in > y_n$ eventually. Choose $0 < \in < \dfrac{l(1-b-c)}{b+c}$.

Now

$$\begin{aligned} x_n &= y_n - b_n x_{n-\tau_1} - c_n x_{n+\tau_2} \\ &> l-(b+c)\, z_{n-\tau_1} \\ &> l-(b+c)(l+\in) \\ &= k(l+\in) > k y_n, \end{aligned}$$

where $k = \dfrac{l-(b+c)(l+\in)}{(l+\in)} > 0.$

Using the above inequality in (2.1), we obtain

$$\Delta\left(a_n\left(d_n\Delta y_n\right)\right) \le -q_n k^{\beta} y_{n+1-\sigma_1}^{\beta} - p_n k^{\beta} y_{n+1+\sigma_2}^{\beta} \le -k^{\beta}\left(q_n+p_n\right) y_{n+1-\tau_1}^{\beta}.$$

Summing the last inequality from n to ∞, we get

$$-\Delta\left(d_n\Delta y_n\right) \le (-kl)^{\beta}\left[\frac{1}{a_n}\sum_{s=n}^{\infty}\left(q_s+p_s\right)\right],$$

which implies

$$\Delta\left(d_n\Delta y_n\right) \ge (kl)^{\beta}\left[\frac{1}{a_n}\sum_{s=n}^{\infty}\left(q_s+p_s\right)\right].$$

Summing again from n to ∞, we obtain

$$-d_n\Delta y_n \ge (kl)^{\beta}\sum_{s=n}^{\infty}\frac{1}{a_s}\sum_{t=s}^{\infty}\left(q_t+p_t\right).$$

This implies

$$-\Delta y_n \ge (kl)^{\beta}\frac{1}{d_n}\sum_{s=n}^{\infty}\left(\frac{1}{a_s}\sum_{t=s}^{\infty}\left(q_t+p_t\right)\right).$$

Summing the above inequality from n_1 to ∞, we have

$$y_n \ge (kl)^{\beta}\sum_{n=n_1}^{\infty}\left(\frac{1}{d_n}\sum_{s=n}^{\infty}\left(\frac{1}{a_s}\sum_{t=s}^{\infty}\left(q_t+p_t\right)\right)\right),$$

which contradicts (2.3). Therefore, $l = 0$.

Also the inequality $0 \le x_n \le y_n$. This implies $\lim_{n\to\infty} x_n = 0$ and hence the proof .

Theorem: 2.5

Assume that condition (2.3) holds, $\sigma_1 \ge \tau_1$ and $\beta \ge 1$. If there exists a positive real sequence $\{\rho_n\}$ and an integer $N_1 \in N$ with

$$\lim_{n\to\infty}\sup\sum_{s=N_1}^{n-1}\left[\rho_s\eta_s\frac{d_{s-\sigma_1}}{d_{s+1-\sigma_1}}-\frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)a_{s-\sigma_1}\left(\Delta\rho_s\right)^2}{4\rho_s}\right]=\infty \quad (2.4)$$

holds, then every solution $\{x_n\}$ of equation (1.1) oscillates or $\lim_{n\to\infty} x_n=0$.

Proof:

Let $\{x_n\}$ be a non-oscillatory solution of equation (1.1). Without loss of generality, wemay assume that there exists an integer $N\ge n_0$ such that $x_n>0,\ x_{n-\sigma_1}>0,$ $x_{n+\sigma_2}>0,\ x_{n-\tau_1}>0,\ x_{n+\tau_2}>0$ for all $n\ge N$. Then $y_n>0$ and (2.1) holds for all $n\ge N$. From (1.1) for all $n\ge N$, we have

$$\begin{aligned}&\Delta(a_n\Delta(d_n\Delta y_n))+q_nx^\beta_{n+1-\sigma_1}+p_nx^\beta_{n+1+\sigma_2}+b^\beta\Delta(a_{n-\tau_1}\Delta(d_{n-\tau_1}\Delta y_{n-\tau_1}))\\&+b^\beta q_{n-\tau_1}x^\beta_{n+1-\tau_1-\sigma_1}+b^\beta p_{n-\tau_1}x^\beta_{n+1-\tau_1+\sigma_2}+\frac{c^\beta}{2^{\beta-1}}\Delta(a_{n+\tau_2}\Delta(d_{n+\tau_2}\Delta y_{n+\tau_2}))\quad(2.5)\\&+\frac{c^\beta}{2^{\beta-1}}q_{n+\tau_2}x^\beta_{n+1+\tau_2-\sigma_1}+\frac{c^\beta}{2^{\beta-1}}p_{n+\tau_2}x^\beta_{n+1+\tau_2+\sigma_2}=0.\end{aligned}$$

Using Lemma 2.1 in (2.5), we have

$$\begin{aligned}&\Delta(a_n\Delta(d_n\Delta y_n))+b^\beta\Delta(a_{n-\tau_1}\Delta(d_{n-\tau_1}\Delta y_{n-\tau_1}))+\frac{c^\beta}{2^{\beta-1}}\Delta(a_{n+\tau_2}\Delta(d_{n+\tau_2}\Delta y_{n+\tau_2}))\\&+\frac{Q_n}{4^{\beta-1}}z^\beta_{n+1-\sigma_1}+\frac{P_n}{4^{\beta-1}}z^\beta_{n+1+\sigma_2}\le 0. \quad (2.6)\end{aligned}$$

By Lemma 2.2, there are two cases for $\{y_n\}$. Assume case (i) holds for $n\ge N_1\ge N$. Since $\Delta y_n>0$, we have $y_{n+\sigma_2}\ge y_{n-\sigma_1}$. Therefore, from (2.6), we have

$$\begin{aligned}&\Delta(a_n\Delta(d_n\Delta y_n))+b^\beta\Delta(a_{n-\tau_1}\Delta(d_{n-\tau_1}\Delta y_{n-\tau_1}))\\&+\frac{c^\beta}{2^{\beta-1}}\Delta(a_{n+\tau_2}\Delta(d_{n+\tau_2}\Delta y_{n+\tau_2}))+\frac{R_n}{4^{\beta-1}}y^\beta_{n+1-\sigma_1}\le 0. \quad (2.7)\end{aligned}$$

Define

$$w_1(n)=\rho_n\frac{a_n\Delta(d_n\Delta y_n)}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}},\quad \text{for } n\ge N_1. \quad (2.8)$$

Then $w_1(n)>0$ for $n\ge N_1$. From (2.8), we can see that

$$\Delta w_1(n)=\frac{\Delta\rho_n}{\rho_{n+1}}w_1(n+1)+\rho_n\frac{\Delta(a_n\Delta(d_n\Delta y_n))}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}}-w_1(n+1)\frac{\rho_n}{\rho_{n+1}}\frac{\Delta(d_{n-\sigma_1}\Delta y_{n-\sigma_1})}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}}.$$

By (2.1), we have $a_{n-\sigma_1}\Delta(d_{n-\sigma_1}\Delta y_{n-\sigma_1})\ge a_{n+1}\Delta(d_{n+1}\Delta y_{n+1})$. Therefore, from (2.8), we get

$$\Delta w_1(n)\le\frac{\Delta\rho_n}{\rho_{n+1}}w_1(n+1)+\rho_n\frac{\Delta(a_n\Delta(d_n\Delta y_n))}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}}-\frac{\rho_n}{\rho^2_{n+1}}\frac{w_1^2(n+1)}{a_{n-\sigma_1}}. \quad (2.9)$$

Next, we define

$$w_2(n)=\rho_n\frac{a_{n-\tau_1}\Delta(d_{n-\tau_1}\Delta y_{n-\tau_1})}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}},\quad \text{for } n\ge N_1. \quad (2.10)$$

Then $w_2(n)>0$ for $n\ge N_1$. Note that $\sigma_1\ge\tau_1$.

Also from (2.1), we find that $a_{n-\sigma_1}\Delta\left(d_{n-\sigma_1}\Delta y_{n-\sigma_1}\right)\geq a_{n+1-\tau_1}\Delta\left(d_{n+1-\tau_1}\Delta y_{n+1-\tau_1}\right)$.
Then from (2.10), we have

$$\Delta w_2(n)\leq\frac{\Delta\rho_n}{\rho_{n+1}}w_2(n+1)+\rho_n\frac{\Delta\left(a_{n-\tau_1}\Delta\left(d_{n-\tau_1}\Delta y_{n-\tau_1}\right)\right)}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}}-\frac{\rho_n}{\rho_{n+1}^2}\frac{w_2^2(n+1)}{a_{n-\sigma_1}}. \qquad (2.11)$$

Also we define

$$w_3(n)=\rho_n\frac{a_{n+\tau_2}\Delta\left(d_{n+\tau_2}\Delta y_{n+\tau_2}\right)}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}},\quad \text{for } n\geq N_1. \qquad (2.12)$$

Then $w_3(n)>0$ for $n\geq N_1$.
By (2.1), we get $a_{n-\sigma_1}\Delta\left(d_{n-\sigma_1}\Delta y_{n-\sigma_1}\right)\geq a_{n+1+\tau_2}\Delta\left(d_{n+1+\tau_2}\Delta y_{n+1+\tau_2}\right)$.

From (2.12), we can find that

$$\Delta w_3(n)\leq\frac{\Delta\rho_n}{\rho_{n+1}}w_3(n+1)+\rho_n\frac{\Delta\left(a_{n+1+\tau_2}\Delta\left(d_{n+1+\tau_2}\Delta y_{n+1+\tau_2}\right)\right)}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}}-\frac{\rho_n}{\rho_{n+1}^2}\frac{w_3^2(n+1)}{a_{n-\sigma_1}}. \qquad (2.13)$$

Therefore, (2.9), (2.11) and (2.13) imply that

$$\begin{aligned}\Delta w_1(n)+b^\beta\Delta w_2(n)+\frac{c^\beta}{2^{\beta-1}}\Delta w_3(n)\leq -\rho_n\frac{R_n}{4^{\beta-1}}\frac{y_{n+1-\sigma_1}^\beta}{d_{n-\sigma_1}\Delta y_{n-\sigma_1}}&\\ +\left(\frac{\Delta\rho_n}{\rho_{n+1}}w_1(n+1)-\frac{\rho_n}{\rho_{n+1}^2}\frac{w_1^2(n+1)}{a_{n-\sigma_1}}\right)&\\ +b^\beta\left(\frac{\Delta\rho_n}{\rho_{n+1}}w_2(n+1)-\frac{\rho_n}{\rho_{n+1}^2}\frac{w_2^2(n+1)}{a_{n-\sigma_1}}\right)&\qquad (2.14)\\ +\frac{c^\beta}{2^{\beta-1}}\left(\frac{\Delta\rho_n}{\rho_{n+1}}w_3(n+1)-\frac{\rho_n}{\rho_{n+1}^2}\frac{w_3^2(n+1)}{a_{n-\sigma_1}}\right)&\end{aligned}$$

Since $\{a_n\}$ is non-decreasing and $\Delta^2 y_n>0$ for $n\geq N_1$, we have $\Delta^3 y_n\leq 0$ for $n\geq N_1$.
Then by Lemma 2.3, we find for any $k\in(0,1)$ and n for sufficiently large

$$\frac{y_{n+1-\sigma_1}}{\Delta y_{n-\sigma_1}}\geq\frac{k(n-\sigma_1)}{2}\frac{d_{n-\sigma_1}}{d_{n+1-\sigma_1}}\quad \text{(by 2.2)} \qquad (2.15)$$

Since $y_n>0, \Delta y_n<0, \Delta\left(d_n\Delta y_n\right)>0$ for $n\geq N_1$, we have

$$y_n=y_{N_1}+\sum_{s=N_1}^{n-1}\Delta y_s\geq(n-N_1)\Delta y_{N_1}\geq\frac{l\,n}{2}, \qquad (2.16)$$

for some $l>0$ and n for sufficiently large. From (2.15), (2.16) and $\beta\geq 1$, we have

$$\frac{y_{n+1-\sigma_1}^\beta}{\Delta y_{n-\sigma_1}}\geq\frac{l^{\beta-1}(n-\sigma_1)}{2^\beta}\frac{d_{n-\sigma_1}}{d_{n+1-\sigma_1}}. \qquad (2.17)$$

Combining the inequality (2.17) with (2.14) and summing the resulting inequality from $N_2\geq N_1$ to $n-1$, we obtain

$$\sum_{s=N_1}^{n-1}\left[\rho_s\eta_s\frac{d_{s-\sigma_1}}{d_{s+1-\sigma_1}}-\frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)a_{s-\sigma_1}\left(\Delta\rho_s\right)^2}{4\rho_s}\right]\le w_1\left(N_2\right)+b^\beta w_2\left(N_2\right)+\frac{c^\beta}{2^{\beta-1}}w_3\left(N_2\right)$$

Taking lim sup for the above inequality, we get a contradiction to (2.4).

Assume that Lemma 2.2(ii) holds. Then by Lemma 2.4, we can obtain $\lim_{n\to\infty}x_n=0$. Hence the proof is complete.

Let $\rho_n=n$ and $\beta=1$. Then from Theorem 2.5, we obtain the following corollary.

Corollary 2.6.
Assume that condition (2.3) holds and $\sigma_1\ge\tau_1$. If there is an integer $N_1\in\mathrm{N}$ with

$$\lim_{n\to\infty}\sup\sum_{s=N_1}^{n-1}\left[s\eta_s\frac{d_{s-\sigma_1}}{d_{s+1-\sigma_1}}-\frac{\left(1+b+c\right)a_{s-\sigma_1}}{4s}\right]=\infty$$

holds, then every solution $\{x_n\}$ of the equation (1.1) oscillates or $\lim_{n\to\infty}x_n=0$.

Theorem 2.7.
Assume that condition (2.3) holds, $\sigma_1\le\tau_1$ and $\beta\ge1$. If there exists a positive real sequence $\{x_n\}$ and an integer $N_1\in\mathrm{N}$ with

$$\lim_{n\to\infty}\sup\sum_{s=N_1}^{n-1}\left[\rho_s\eta_s\frac{d_{s-\tau_1}}{d_{s+1-\tau_1}}-\frac{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)a_{s-\sigma_1}\left(\Delta\rho_s\right)^2}{4\rho_s}\right]=\infty,$$

holds, then every solution $\{x_n\}$ of the equation (1.1) oscillates or $\lim_{n\to\infty}x_n=0$.

Proof.
Proceeding as in the proof of Theorem 2.5, we get (2.6). Assume Lemma 2(i) holds for all $n\ge N_1\ge N$. Then we obtain (2.7). Now consider the following transformations

$$w_1\left(n\right)=\rho_n\frac{a_n\Delta\left(d_n\Delta y_n\right)}{d_{n-\tau_1}\Delta y_{n-\tau_1}},\quad\text{for } n\ge N_1.$$

$$w_2\left(n\right)=\rho_n\frac{a_{n-\tau_1}\Delta\left(d_{n-\tau_1}\Delta y_{n-\tau_1}\right)}{d_{n-\tau_1}\Delta y_{n-\tau_1}},\quad\text{for } n\ge N_1.$$

$$w_3\left(n\right)=\rho_n\frac{a_{n+\tau_2}\Delta\left(d_{n+\tau_2}\Delta y_{n+\tau_2}\right)}{d_{n-\tau_1}\Delta y_{n-\tau_1}},\quad\text{for } n\ge N_1.$$

and as in the proof of Theorem 2.5, we can get

$$\Delta w_1(n)+b^{\beta}\Delta w_2(n)+\frac{c^{\beta}}{2^{\beta-1}}\Delta w_3(n)\le -\rho_n\frac{R_n}{4^{\beta-1}}\frac{y^{\beta}_{n+1-\tau_1}}{d_{n-\tau_1}\Delta y_{n-\tau_1}}$$
$$+\left(\frac{\Delta\rho_n}{\rho_{n+1}}w_1(n+1)-\frac{\rho_n}{\rho^2_{n+1}}\frac{w_1^2(n+1)}{a_{n-\tau_1}}\right)$$
$$+b^{\beta}\left(\frac{\Delta\rho_n}{\rho_{n+1}}w_2(n+1)-\frac{\rho_n}{\rho^2_{n+1}}\frac{w_2^2(n+1)}{a_{n-\tau_1}}\right) \qquad (2.19)$$

By Lemma 2.3, for any $k\in(0,1)$, we find

$$\frac{y_{n+1-\sigma_1}}{\Delta y_{n-\tau_1}}\ge\frac{k(n-\tau_1)}{2}\frac{d_{n-\tau_1}}{d_{n+1-\tau_1}}$$

and $\Delta(d_n\Delta y_n)>0$ for $n\ge N_2$. Then proceeding as in the proof of Theorem 2.1, we get

$$\sum_{s=N_2}^{n-1}\left[\rho_s\eta_s\frac{d_{s-\tau_1}}{d_{s+1-\tau_1}}-\frac{\left(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}}\right)a_{s-\sigma_1}(\Delta\rho_s)^2}{4\rho_s}\right]\le w_1(N_2)+b^{\beta}w_2(N_2)+\frac{c^{\beta}}{2^{\beta-1}}w_3(N_2).$$

Taking lim sup on both sides of the last inequality, we obtain a contradiction with (2.18).
Assume that case(ii) holds. Then by Lemma 2.4, we obain $\lim_{n\to\infty}x_n=0$ and hence the proof.

Let $\rho_n=n$ and $\beta=1$. Then we get the following corollary.

Corollary 2.8.
Assume that condition (2.3) holds and $\sigma_1\le\tau_1$. If

$$\limsup_{n\to\infty}\sum_{s=N}^{n-1}\left[s\eta_s\frac{d_{s-\tau_1}}{d_{s+1-\tau_1}}-\frac{(1+b+c)a_{s-\tau_1}}{4s}\right]=\infty$$

holds for all sufficiently large N, then every solution $\{x_n\}$ of the equation (1.1) oscillates or $\lim_{n\to\infty}x_n=0$.

III. Example

Example 3.1.
Consider the third order difference equation

$$\Delta^3\left(x_n+\frac{1}{4}x_n+\frac{1}{4}x_{n+1}\right)+\left(\frac{16}{3}\right)9^n x^3_{n+1}+(144)9^n x^3_{n+2}=0. \qquad (3.1)$$

Let $a_n=d_n=1,\ b_n=c_n=\frac{1}{4},\ q_n=\left(\frac{16}{3}\right)9^n,\ \ p_n=(144)9^n$

and $\tau_1=0,\tau_2=1,\ \sigma_1=0,\sigma_2=1$.
Then condition (2.3) holds and condition (2.4) also holds. Therefore all conditions of Theorem 2.5 hold, and hence every solution of equation (3.1) is oscillatory or tends to zero as $n\to\infty$. One such solution is $x_n=\frac{1}{3^n}$.

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