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# On Scalar Pseudo Commutativity of Algebras over a Commutative Ring

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***Abstract:*** *The concept of scalar commutativity defined in an algebra over a ring is mixed with the concept of pseudo commutativity defined in a near – ring to define the new concept of scalar pseudo commutativity in an algebra over a ring and many interesting results are obtained.*

## I. Introduction

Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar commutative if for each x,y ϵ A, there exists α ϵR depending on x and y such that xy = αxy. Rich [8] proved that if A is scalar commutative over a field F, then A is either commutative or anti – commutative. Koh, Luh and Putcha [6] proved that if A is scalar commutative with identity 1 and if R is a Principal ideal domain, then A is commutative. A near ring N is said to be pseudo commutative [9] if xyz =zyx for all x,y,z ϵ N. In this paper we define scalar pseudo commutativity in an algebra A over a commutative ring R and prove many interesting results.

## II. Preliminaries

**2.1 Definition [9]**
Let N be a near ring. N is said to be pseudo commutative if xyz =zyx for all x,y,z ϵ N.
**2.2Definition**
Let N be a near ring N is said to be pseudo anti – commutative if xyz = -zyx for all x,y,z ϵ N.
**2.3 Definition [8]**
Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar commutative if for each x,y ϵ A, there exists a scalar α = α(x,y) ϵR depending on x and y such that xy = αxy. It is said to be scalar anti – commutative if xy = - αyx.
**2.4 Lemma [5]**
Let N be a distributive near – ring. If xyz = ± zyx for all x,y,z ϵ N, then N is either pseudo commutative or pseudo anti – commutative.

## III. Main Results

**3.1 Definition**
Let A be an algebra over a commutative ring R. A is said to be scalar pseudo commutative if for every x,y,z ϵ A, there exists a scalar α = α(x,y,z) ϵ R depending on x,y,z ϵ A such that xyz = αzyx.. It is said to be scalar pseudo anti – commutative if xyz = - αzyx.

**3.2 THEOREM:**
Let A be an algebra (not necessarily associative) over a field F. If A is scalar pseudo commutative, then A is either pseudo commutative or pseudo anti-commutative.

**Proof:**
Suppose xyz = zyx for all x,y,z ϵ A, there is nothing to prove.
Suppose not, we will prove that xyz = -zyx for all x,y,z ϵ A,
We shall first prove that if x,y,z ϵ A such that xyz ≠ zyx, then xyx = zyz = 0
Let x,y,z ϵ A such that xyz ≠ zyx .
Since A is scalar pseudo commutative, there exist scalars α = α (x,y,z) ϵ F and β = β (x+z, y, z) ϵ F such that

$$xyz = \alpha\, zyx \quad \ldots\ldots (1)$$
$$(x+z)\, yz = \beta\, zy(x+z) \quad \ldots\ldots (2)$$

(1) - (2) gives

$$xyz - xyz - zyz = \alpha zyx - \beta zyx - \beta zyz$$
$$(\beta - 1)\, zyz = (\alpha - \beta)\, zyx \quad \ldots\ldots (3)$$

Now zyx $\neq 0$ for if zyx = 0 then from (1)
xyz = 0 and so xyz = zyx, a contradiction to our assumption that xyz $\neq$ zyx.
Also $\beta \neq 1$ for if $\beta = 1$, then from (3) we get $\alpha - \beta = 0$. Hence $\alpha = \beta = 1$.
Then from (1) we get xyz = zyx, again a contradiction.
From (3), we get, zyz = $\frac{\alpha-\beta}{\beta-1}$ zyx
That is, zyz = $\gamma$zyx for some $\gamma \in$ F ………………………(4)
Similarly xyx = $\delta$ zyx for some $\delta \in$ F ……………………………..(5)
Now, corresponding to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F$, there exists $\eta \in$ F such that $(\alpha_1 x + \alpha_2 z)\ y\ (\alpha_3 x + \alpha_4 z) = \eta\ (\alpha_3 x + \alpha_4 z)\ y\ (\alpha_1 x + \alpha_2 z)$
$(\alpha_1 xy + \alpha_2 z y)\ (\alpha_3 x + \alpha_4 z) = \eta\ (\alpha_3 x\, y + \alpha_4 z\, y)\ (\alpha_1 x + \alpha_2 z)$

$\alpha_1\ \alpha_3$ xyx + $\alpha_1 \alpha_4$ xyz + $\alpha_2 \alpha_3$ zyx + $\alpha_2\alpha_4$ zyz = $\eta$ ($\alpha_3\ \alpha_1$ xyx + $\alpha_3 \alpha_2$ xyz + $\alpha_4 \alpha_1$ zyx + $\alpha_4\alpha_2$ zyz) ………(6)

$\alpha_1\ \alpha_3\ \delta$zyx + $\alpha_1 \alpha_4$ xyz + $\alpha_2 \alpha_3$ zyx + $\alpha_2\alpha_4\ \gamma$zyx
= $\eta$ ($\alpha_3\ \alpha_1\ \delta$zyx + $\alpha_3 \alpha_2$xyz + $\alpha_4 \alpha_1$zyx + $\alpha_4\alpha_2\ \gamma$zyx ) (using (4) and (5))
$(\alpha_1\ \alpha_3 \delta\ \alpha^{-1} + \alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1} + \alpha_2\ \alpha_4\ \gamma\ \alpha^{-1})$ xyz
= $\eta\ (\alpha_3\ \alpha_1 \delta + \alpha_3 \alpha_2 \alpha + \alpha_4 \alpha_1 + \alpha_4\alpha_2\ \gamma)$ zyx (using (1))
Taking $\alpha_3 = 0$, $\alpha_4 = \alpha_2 = 1$, $\alpha_1 = -\gamma$, the RHS of (6) is Zero. Where as the LHS of (6) becomes
$(-\gamma + \gamma\ \alpha^{-1})$ xyz = 0
Ie., $\gamma\ (\alpha^{-1} - 1)$ xyz = 0
Since xyz $\neq$ 0 and $\alpha \neq 1$, We get $\gamma = 0$.
Hence from (4), we get zyz = 0 ………………(7)
Also taking $\alpha_2 = 0$, $\alpha_3 = \alpha_1 = 1$, $\alpha_4 = -\delta$, the RHS of (6) is Zero. Whereas the LHS of (6) becomes
$(\delta\ \alpha^{-1} - \delta)$ xyz = 0
ie., $\delta\ (\alpha^{-1} - 1)$ xyz = 0
Since xyz $\neq$ 0 and $\alpha \neq 1$, We get $\delta = 0$.
Hence from (5), we get xyx = 0 ………………(8)
Now (6) becomes,
$\alpha_1 \alpha_4$xyz + $\alpha_2 \alpha_3$zyx = $\eta$ ( $\alpha_3 \alpha_2$ xyz + $\alpha_4 \alpha_1$ zyx)
$\alpha_1 \alpha_4$ xyz + $\alpha_2 \alpha_3 \alpha^{-1}$ xyz = $\eta$ ($\alpha_3 \alpha_2$xyz + $\alpha_4 \alpha_1\ \alpha^{-1}$ xyz) Using (1)
$(\alpha_1 \alpha_4 + \alpha_2 \alpha_3 \alpha^{-1})$ xyz = $\eta\ (\alpha_3 \alpha_2 + \alpha_4 \alpha_1\ \alpha^{-1})$ xyz …………………….(9)
This is true for all choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in$ F .
Taking $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\alpha_2 = -\alpha^{-1}$ the RHS of (9) is Zero.
The LHS of (9) becomes
$(1 - (\alpha^{-1})^2)$ xyz = 0
Since xyz $\neq$ 0 , $1 - (\alpha^{-1})^2 = 0$. Hence $\alpha = \pm 1$
Since $\alpha \neq 1$ , we get $\alpha = -1$.
Hence xyz = - zyx for x, y, z $\in$ A
Thus A is either Pseudo commutative or Pseudo anti commutative.

**3.3 Lemma**
Let A be an algebra (not necessarily associative) over a commutative ring R. Suppose A is scalar pseudo commutative. Then for all x, y, z $\in$ A, $\alpha \in$ R, $\alpha$xyz = 0 iff $\alpha$zyx = 0 .Also xyz = 0 iff zyx = 0

**Proof:**
Let x, y, z $\in$ A and $\alpha \in$ R such that $\alpha$xyz = 0 . Since A is scalar pseudo commutative there exists $\beta = \beta$ (z, y, $\alpha$x) $\in$ R such that
zy($\alpha$x) = $\beta$ ($\alpha$x) yz = $\beta\alpha$xyz = 0
ie. $\alpha$zyx = 0
Similarly if $\alpha$zyx = 0, then there exists $\gamma = \gamma$ ($\alpha$x , y, z) $\in$ R such that
$\alpha$xyz = $\gamma$zy($\alpha$x) = $\gamma\alpha$zyx = 0
Thus $\alpha$xyz = 0 iff $\alpha$zyx = 0.
Assume xyz = 0. Since A is pseudo commutative there exists
$\delta = \delta$(z, y, x) $\in$ R such that zyx = $\delta$xyz = 0.
Similarly if zyx, there exists $\gamma = \gamma$ (x, y, z) such that xyz = $\gamma$ zyx = 0
Then xyz = 0 iff zyx = 0.

**3.4 LEMMA :**
Let A be an algebra over a commutative ring R. Suppose A is scalar pseudo commutative. Let x, y, z, u ϵ A, α , β ϵ R such that uyx = xyu, and zyx = αxyz and ( z + u) yx = β xy (z+u). Then ( u – αu)y (x-βx) = 0.

**Proof:**

Let x, y, z, u ϵ A
Given zyx = αxyz ………………(1)
( z + u) yx = β xy (z+u) …………..(2)
uyx = xyu …………..(3)

From (2) , we get
zyx + uyx = βxyz + βxyu
αxyz + uyx = βxyz + βxyu (using (1))
αxyz + xyu =` βxyz + βxyu (using (3))
xy ( αz + u – βz – βu ) = 0

By lemma 3.3, we get
( αz + u – βz – βu ) yx = 0
αzyx + uyx – βzyx – βuyx = 0
αzyx + uyx –α βxyz – βuyx = 0 ……………(4)

From (2) , we get
zyx + uyx – βxyz – βxyu = 0

Multiplying by α
α zyx + α uyx – α βxyz – α βxyu = 0 ………………(5)

From (4) and (5) , we get
uyx – β uyx – α uyx + α β xyu = 0
uyx – αuyx – β uyx + αβuyx = 0 (using( 3) )
(u – αu) yx – (u- αu) βyx = 0
(u – αu) (yx – βyx ) = 0
(u – αu) y (x – βx ) = 0
Hence proved.

**3.5 Corollary:**

Taking u = x, we get
(x – αx) y (x – βx ) = 0

**3.6 Lemma:**
Let A be an algebra over a commutative ring R. Suppose A has no zero divisors . If A is scalar pseudo commutative, then A is pseudo commutative.

**Proof:**
Let x, y, z ϵ A, since A is scalar pseudo commutative, there exists scalars
α = α (z,y,x) ϵ R and β = β (z+x, y, x) ϵ R such that

zyx = α xyz ……………….(1)
and ( z + x) yx = β xy (z+x)………………(2)

From (2), we get
zyx + xyx = βxyz + βxyx
αxyz + xyx = βxyz + βxyx (using (1))
xy ( αz + x – βz – βx ) = 0

By lemma 3.3, we get
( αz + x – βz – βx ) yx = 0
αzyx + xyx – βzyx – βxyx = 0
αzyx + xyx –α βxyz – βxyx = 0 …………….(3)

Also from (2) , we get
zyx +xyx = βxyz + βxyx

Multiplying by α
α zyx + α xyx - α βxyz - α βxyx = 0
α zyx - α β xyx = α βxyz -α xyx ………………(4)

From (3) and (4) , we get
xyx – β xyx + α β xyx - α xyx = 0
xyx – αxyx – β xyx + αβxyx = 0

$(x - \alpha x)\ yx - \beta\ (x - \alpha x)\ yx = 0$
$(x - \alpha x)\ yx - (x - \alpha x)\ \beta\ yx = 0$

i.e, $(x - \alpha x)\ (yx - \beta yx) = 0$
i.e, $(x - \alpha x)\ y\ (x - \beta x) = 0$ ………………..(5)

Since A has no zero divisors, $x = \alpha x$ or $x = \beta x$.
If $x = \alpha x$, then from (1), we get $zyx = xyz$
If $x = \beta x$, then from (2) , we get

$(z+x)\ yx = xy\ (z+x)$
$zyx+xyx = xyz+xyx$
ie., $zyx = xyz$

Thus A is pseudo commutative.
Hence proved.

**3.7 Definition:**
Let R be any ring and x, y, z $\epsilon$ R. We define $xyz - zyx$ as the pseudo commutator of x, y, z.

**3.8 Theorem:**
Let A be an algebra over a commutative of ring R. Let A be scalar pseudo commutative. If A has an identity, then the square of every pseudo commutator is zero ie., $(xyz - zyx)^2 = 0$ for all x, y, z $\epsilon$ A .

**Proof:**
Let x, y, z $\epsilon$ A . since A is pseudo commutative , there exists scalars
$\alpha = \alpha(z,y,1)\ \epsilon$ R and $\beta = (z+1, y, 1)\ \epsilon$ R such that

$zy.1 = \alpha 1.yz$
$zy = \alpha yz$ ……………….(1)

and $(z+1)\ y.1 = \beta 1.y(z+1)$
$(z+1)\ y = \beta\ y(z+1)$ …………….(2)

From (2), we get

$zy + y = \beta yz + \beta y$
$\alpha yz + y - \beta yz - \beta y = 0$ (using(1))
$1.y(\alpha z+1-\beta z-\beta) = 0$

Hence proved.

**3.9 Definition:**
Let R be a P.I.D and A be an algebra over R. Let a $\epsilon$ A. Then the order of a denoted as O(a) is defined to be the generator of the ideal $I = \{\alpha \epsilon R / \alpha a = 0\}$. O(a) is unique upto associates and O(a) = 1 if and only if a = 0.

**3.10 Lemma :**
Let A be an algebra with identity over a principal ideal domain R. If A is scalar pseudo commutative, y $\epsilon$ R and O(y) = 0, then y is in the center of A.

**Proof :**
Let y $\epsilon$ A such that O(y) = 0. Let x $\epsilon$ A be any element.
Now there exist scalars $\alpha = \alpha\ (1,y,x)\ \epsilon$ R and $\beta = \beta(x+1, y, 1)\ \epsilon$ R such that $1.\ yx = \alpha xy = 1$. That is $yx = \alpha xy$ ……….(1)
$(x+1)y.1 = \beta.1.y(x+1)$. That is $(x+1)y = \beta\ y(x+1)$ ……….(2)
From (2) weget

$xy+y- \beta yx- \beta y = 0$
$xy+y-\alpha\beta xy-\beta y = 0$ (using (1))
$(x+1-\alpha\beta x-\beta)y.1 = 0$

By Lemma 3.3 ,we get

$1.y\ (x+1-\alpha\beta x- \beta) = 0$
$yx+y-\alpha\ \beta yx-\beta y = 0$ …………..(3)

Also from (2) weget

$xy + y - \beta yx-\beta y = 0$

Multiply by $\alpha$

$\alpha\ xy+ \alpha y- \alpha\ \beta yx- \alpha\ \beta y = 0$
$yx+ \alpha y- \alpha\ \beta yx- \alpha\ \beta y = 0$ …………………(4)

From (3) and (4), we get

$y - \beta y - \alpha y + \alpha\beta y = 0$
$y(1-\beta) - \alpha(1-\beta) y = 0$
Ie., $(1-\beta)(y-\alpha y) = 0$
$(1-\beta) y (1-\alpha) = 0$
Since $O(y) = 0$, we get $(1-\alpha) = 0$ or $(1-\beta) = 0$
Ie., $\alpha = 1$ or $\beta = 1$
If $\alpha = 1$, from (1), we get $yx = xy$
If $\beta = 1$, from (2), we get $(x+1)y = y(x+1)$
$xy + y = yx+y$
$xy = yx$
Thus y commutes with every $x \in A$.
Hence y belongs to the center of A.

**3.11 Lemma:**
Let A be an algebra with unity over a P.I.D R . If A is scalar pseudo commutative , $y \in A$ such that $O(y) = 0$, then $xyz = zyx$ for all $y, z \in A$.
**Proof:**
Let $y \in A$ with $O(y) = 0$
For $x, z \in A$, there exists scalars $\alpha = \alpha(z, y, x) \in R$ and $\beta = \beta(x+1, y,z) \in R$ such that $zyx = \alpha xyz$ ………… (1)
$(x+1)yz = \beta zy(x+1)$ …………..(2)
From (2), we get
$xyz + yz = \beta zyx + \beta zy$
$= \alpha\beta xyz + \beta zy$
$= \alpha\beta xyz + \beta yz$ (using Lemma 3.10)
$xyz + yz - \alpha\beta xyz - \beta yz = 0$
$(x+1 - \alpha\beta x - \beta) yz = 0$
ie., $zy(x+1-\alpha\beta x - \beta) = 0$
ie., $zyx + zy - \alpha\beta zyx - \beta zy = 0$ …………….(3)
Also from (2), we get
$xyz + yz - \beta zyx - \beta zy= 0$
Multiplying $\alpha$
$\alpha xyz + \alpha yz - \alpha\beta zyx - \alpha\beta zy= 0$
$zyx + \alpha yz - \alpha\beta zyx - \alpha\beta zy= 0$ (using (1))……………..(4)
From (3) and (4), we get
$zy - \beta zy + \alpha yz - \alpha\beta zy = 0$
$yz - \beta yz + \alpha yz - \alpha\beta yz = 0$ (since $O(y) = 0$ using Lemma 3.10)
$(1 - \beta - \alpha + \alpha\beta) yz = 0$
$(1-\alpha)(1-\beta) yz = 0$ for all $z \in A$ ……………….(5)
Thus for each $z \in A$, there exists scalars $\gamma \in R$, $\delta \in R$ such that
$\gamma y z = 0$ …………….(6) and
$\delta y (z+1) = 0$ …………. (7)
$\delta y z + \delta y = 0$
Multiplying by $\gamma$
$\gamma \delta y z + \gamma \delta y = 0$ ………….(8)
From (6), we get $\gamma \delta y z = 0$ ………………..(9)
(8) and (9) gives
$\gamma \delta y = 0$ . Since $O(y) = 0$, we get $\gamma = 0$ or $\delta = 0$
Hence from (5), we get $1-\alpha = 0$ or $1-\beta = 0$
Then $\alpha = 1$ or $\beta = 1$
If $\alpha = 1$, from (1) we get, $zyx = xyz$
If $\beta =1$, from (2) we get
$(x+1) yz = zyx + zy$
$Xyz + yz = zyx + zy$
$xyz + yz = zyx + yz$ (using Lemma 3.7)
$xyz = zyx$
Hence A is pseudo commutative

**3.12 Lemma:**
Let A be an algebra with identity over a commutative ring R. Then
(i) A is scalar pseudo commutative iff A is scalar weak commutative
(ii) A is scalar pseudo commutative iff A is scalar quasi weak commutative
(iii) A is scalar weak commutative iff A is scalar quasi weak commutative

**Proof :**
(i) Assume A is scalar pseudo commutative
Let x, y, z ϵ A
Now xyz = x (yz. 1)
= x(α1zy) for some α = α(y, z, 1) ϵ R
( Since A is scalar pseudo commutative)
= αxzy
Thus A is scalar weak commutative.
Conversly assume A is scalar weak commutative
Then for any x, y, z ϵ A
xyz = x (1.yz)
= x (α1zy) (since A is scalar weak commutative)
= αxzy
= α (1.xz ) y
= α (β1zx) y (since A is scalar weak commutative )
= αβzxy
= αβz (1.xy)
= αβz (γ1.yx) (since A is scalar weak commutative )
= αβγ zyx
xyz = δ zyx for some δ ϵ R
Hence A is scalar pseudo commutative.
The proof o f (ii) and (iii) are straight forward.

**3.13 Lemma:**
Let A be any ring with identity. Then
(i) A is weak commutative iff A is pseudo commutative
(ii) A is pseudo commutative iff A is quasi weak commutative
(iii) A is quasi weak commutative iff A is weak commutative

**Proof :**
(i) Assume A is weak commutative.
Let x, y, z ϵ A
xyz = x (1.yz)
= x(1.zy) (since A is weak commutative)
= (1.xz) y (since A is weak commutative)
= (1.zx)y
= z (1xy)
= z(1yx) (since A is weak commutative)
= zyx
Thus A is weak commutative implies pseudo commutative.
Conversly assume A is pseudo commutative.
Let x, y, z ϵ A
xyz = x (1yz)
= x (zy 1) (since A is pseudo commutative)
= xzy
Thus A is weak commutative
The proof of (ii) and (iii) are straight forward.

**3.14 Lemma:**
Let A be an algebra with identity over a P.I.D R. Suppose that A is scalar pseudo commutative. Assume further that there exists a prime p ϵ R and positive integer m ϵ $Z^+$ such that $p^m A = 0$ . Then A is pseudo commutative.

**Proof :**

Let A be an algebra with identity over a commutative ring R.

Then A is scalar pseudo commutative implies A is scalar weak commutative ( By lemma3.12) and so A is weak commutative.

Again A is weak commutative implies A is pseudo commutative.

Hence proved.

**3.15 Theorem :**

Let A be an algebra with identity over a P.I.D R. If A is scalar pseudo commutative, then A is pseudo commutative.

**Proof :**

A is scalar pseudo commutative implies A is scalar weak commutative (Lemma 3.12 (i) )

A is scalar weak commutative implies A is weak commutative.

A is weak commutative implies A is pseudo commutative (Lemma 3.13(i))

Hence proved.

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