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Existence of Limit Cycles in a Simple Predator-Prey Model

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***Abstract**: One of the useful tools and often used in the environmental science eld is the predator-prey models,this is because they allow researchers to both observe the dynamics of animal populations and make predictions as how they will develop over time. In this paper, six projections of animal populations based on a simple predator-prey model were created and explore the trends visible.Starting with a set of initial conditions that produced di erent outcomes for the function of the population of rabbits and foxes over a period of 100years time span.Using Runge Kutta method,a MATLAB program was developed to produce the values for the population of rabbits and foxes over the time span of 100years.*

***Keywords**: predator, prey, limits cycle.*

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I. Introduction

Predator-prey models are well known in the environmental study due to their ability to math-ematically predict animal populations as well as explain trends that can be observed in species population and density[1]. This mapping out of populations is greatly bene cial because it allows for scientists to track ecosystems and biodiversity[3]. By observing the stability of a predator-prey relationship over time one can measure the strength and stability of the ecosys-tem itself. Having the knowledge on the way an ecosystem functions, one can then observe the a ects and changes the human population has on the ecosystems survival[1].Predator-prey models are arguably the building blocks of the bio- and ecosystems as biomasses are grown out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their speci c settings of applications, they can take the forms of resource-consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise[4].One of the rst models to incorporate interactions between predators and prey was proposed in 1925 by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra. The Lotka-Volterra model is based on di er-ential equations[7]. The Lotka-Volterra model is one of the earliest predator-prey models to be based on sound mathematical principles. It forms the basis of many models used today in the analysis of population dynamics[3]. The predator-prey models are the basis which helps to see the life and ecosystems in which the predator and prey have certain role. This model explains how the predators interact with its prey. It explains the sustenance, evaluation and alternative dispersion of some species in the case of failure to complete in the life in which the stronger has advantage of dominate role. The predator-prey model is like the survival of the ttest-theory (Zhang and Liu, 2015)[6]. The ttest are the stronger species targeting the weaker species and win life for themselves and this evaluation of life for one species results in the numerical and sometime general extinction of other weaker species. The weaker species remain in constraint struggle to achieve their security in the diaspora where general fear of life remains ever present. The weaker species which become prey adopt many measures to trick the predator to avoid being hunted[3,2].

An application of the nonlinear system of di erential equations in mathematical biology is to model the predator-prey relationship of a simple ecosystem.Supposed in a closed ecosys-tem,there are only two types of animals,the predator and the prey.They form a simple food chain where the predator species hunts the prey species,while the prey grazes vegetation.The size of the two populations can be described by a simple system of two nonlinear rst order di erential equations i.e the Lotka-Volterra equations which originated in the study of sh population of the Mediterranean during and immediately after world war I(WW I)[2].There are several reasons for studying predaror-prey models. Mathematically, some versions of this models exhibit limit cycles, an important type of equilibrium sometimes observed in dynami-cal systems with two or more dimensions.Predator-prey models have a variety of useful social science applications.

II. Model Formulation

One of the rst two-species ecosystem which has been mathematically modelled involves a predator and its prey[3]. We will study the interaction in the forest between the foxes and small rabbits consumed by the foxes.Before we show the formulation of the model,let us describe the e ect we would like to model.If hunters refrained from hunting for some time, then these animals are expected to increase in number. Once having increased, the foxes would have enough food to sustain a larger population of foxes. Thus,the population of foxes would increase, and in short time pose a severe threat to the small rabbits. Eventually the population of the rabbits would diminished, the foxes can no longer sustain their enlarged population and must decrease in number and in turn allows the rabbits to return towards their original population.

2.1 ASSUMPTIONS

The following are assumed;
The area is assumed bounded or closed ecosystem(migration across the boundary is,if not impossible at least unlikely to be a major factor.
There are only two types of animals,the predator and the prey.
They form a simple food chain where the predator species hunts the prey species,while the prey grazes vegetation.
Prey birth rate is proportional to the size of the prey population.
Prey death rate is proportional to the size of both the predator and prey population.
Predator birth rate is proportional to the sizes of both the predator and prey populations. Predator death rate is proportional to the size of the predator population.

In developing such an ecosystem model that represents the interaction of two species, the foxes and the rabbits, it is of advantage to rst model the population growths ignoring the interactions between the species, that is the equation the population of foxes satisfy in the absence of rabbits and vice versa.
Now let R(t) denote the population of rabbit species at a time t,and F (t) denotes the population of foxes species at time t.

Considering R(t) in the absence of F (t)i:eF (t) = 0, the assumptions concerning the birth and death processes of the rabbits must be outlined. Since the rabbits graze on the presumed abundant vegetation, it is suspected that the growth rate of the rabbits when F (t) = 0 is constant i.e

$$\frac{dR}{dt} = aR \qquad (1)$$

where a is the number of o spring per rabbit per year or the intrinsic rate of increase of rabbit per year, the assumption that the the vegetation is unlimited, the birth rate is larger than the death rate,so when F (t) = 0; R(t) would grow exponentially without bound.i.e

$$R(t) = R_o e^{at} \qquad (2)$$

.On the other hand,when there is presence of foxes i.e F (t) > 0, then a logistic growth model might be proposed as

$$\frac{dR}{dt} = aR \;\; bRF \qquad (3)$$

where b is the proportion of the rabbit population consumed by one fox or a constant predation rate(fraction of the rabbit eaten per fox).
The foxes behave in an entirely di erent manner.If there are no rabbits,then the food source of the foxes is absent.In this case,the death rate of foxes is expected to exceed the birth rate,hence
in the absence of rabbits

$$\frac{dF}{dt} = \;\; dF \qquad (4)$$

where d is a per capita mortality rate of foxes or proportion of foxes population dying per year,so when R(t) = 0,the foxes would be an endangered species eventually die out or their population vanish

$$F(t) = F_o e^{\;\; dt} \qquad (5)$$

We now model the complex interaction between rabbits and foxes. The presence of rabbits enhances the fox population,thus, the rabbit population will cause an increase in the growth rate of the foxes,for this process,we assume that the growth rate of the foxes is increased proportional to the number of rabbits.The growth rate of the foxes without rabbit was modeled as d, but with rabbit is modelled as d + bcR where c is a constant conversion rate of eaten rabbit into new fox abundance or conversion of one rabbit consumed into new foxes. Thus

$$\frac{dF}{dt} = F(\ d + bcR) = \ dF + bcFR \qquad (6)$$

We have described the background to set of di erential equations developed independently by Lotka and Volterra in the 1920s.
Mathematically, the model is conventionally expressed as:

$$\begin{aligned} \frac{dR}{dt} &= aR \ bRF \\ \frac{dF}{dt} &= bcFR \ dF \end{aligned} \qquad (7)$$

Where

R =size of the rabbit population

F =size of the fox population

a =number of o spring per rabbit per year

b =proportion of the rabbit population consumed by one fox per year c =conversion of one rabbit consumed into new foxes

d =proportion of foxes population dying per year.

we now focus on the question whether the qualitative behavior of solutions of this system of di erential equations seem consistent with the normal observed oscillatory behavior.

2.2 EQUILIBRIUM

There are two critical points.The trivial steady state which is the origin $R_i = F_i = 0$ for all i i.e $(R_1; F_1) = (0; 0)$ and the non-trivial $(R_2; F_2) = (\frac{d}{bc}; \frac{a}{b})$ in the rst quadrant.

2.3 STABILITY

From the preceding expressions for the equilibria.The stability of each of the two steady states can be assessed more formally using the following approach. The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and nding the eigenval-ues evaluated at each equilibrium point.For stability of the equilibrium points,the real parts of the eigenvalues of the Jacobian matrix must be negative,the Jacobian matrix of the system
(5) is given by

$$J(E_i) = \begin{pmatrix} a \ bF & bR \\ bcF & bcR \ d \end{pmatrix} \qquad (8)$$

At (0; 0) , the linearized system has coe cient matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \qquad (9)$$

The eigenvalues are a > 0 and d < 0,hence the equilibrium point (0; 0) is unstable saddle point.This steady state is the simplest or the trivial dead-ecosystem state for which the rabbits and the foxes are absent,this situation is unrealistic as it indicates no any interaction between the rabbits and the foxes,hence it will be no more considered.

At $(\frac{d}{bc}; \frac{a}{b})$,the linearized system has coe cient matix

$$A = \begin{pmatrix} 0 & \frac{d}{c} \\ ac & 0 \end{pmatrix} \quad (10)$$

The eigenvalues are

$$\lambda_{1,2} = \sqrt{ad}\, i \quad (11)$$

Thus the eigenvalues are purely imaginary with 0 real parts. This creates a kind of dynamics right on the cusp between stability and instability, called centre or neutral stability.

2.4 PARAMETERS ESTIMATION

There are three factors that in uence a model behavior, the model structure, the math-ematical formulation of the ows and the parameters that describe the strength of the ow functions [3]. The description of the parameter ranges used by some other authors are given below, but variation of parameter values across models is inevitable and direct comparison is not always easy since di erent functional forms are sometimes used, but in many cases the parameter de nitions are the same [9].

Table 2.1: parameter values

Symbol	Description	Value	Unit
a	Birth rate of rabbit	0.2	rabbit/year
b	Birth rate of fox	0.001	rabbit/year
c	Death rate of rabbit	0.001	fox/year
d	Death rate of fox	0.5	fox/year

Table 2.2: Initial Populations

CASES	NO of Rabbits	NO of Foxes
1	50	500
2	500	50
3	250	100
4	520	100
5	520	150
6	500	200

III. Results

We now focus on nding the numerical solution of the two di erential equations using an in built method ODE45 solver in the MATLAB base on the parameter values obtained in Table 2.1.and the initial conditions set for the projections in Table 2.2 , in which the values of R and
F would be plotted against values for time.

IV. Discussion

The objective of this paper is to create six projections of animal populations based on a simple predator-prey model and explore the trends visible.We started each case with a di erent set of initial conditions that in turn indicated how the populations changed over time.
In the rst case, we started with a rabbit population as a prey of 50 and a fox population of 500.

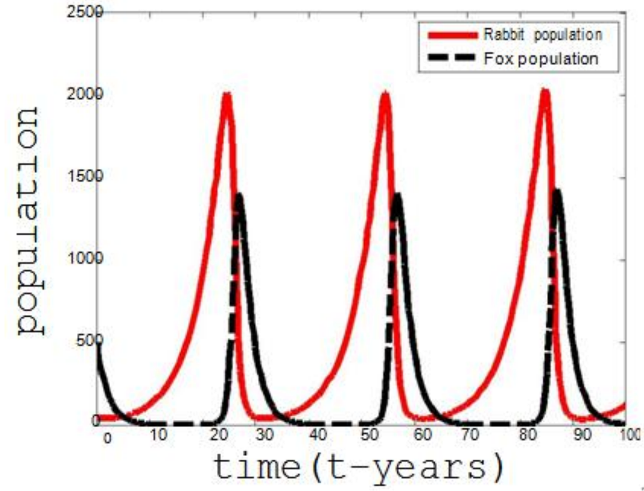

Figure 1: populations of the two species over 100 year span starting with R = 50,F = 500

As we can see from gure 1,the change in population is clearly periodic in nature for this case. We can also notice relationship between the two populations,one can easily see the pre-diction that as the population of the rabbits increases, an increase in the fox population will follow. This trend seems consistent since the population of foxes in the rst order di erential equations is dependent on the population of its food source,the rabbits.The reverse can also be seen that the rabbit population also depends on that of the foxes.It is also seen that the cycle begins with a decline in the fox population and continues until the fox population is very low and correlates with an increase in the rabbit population.Also from the graph,the rise and fall of the populations occurs close to thirty years increments and the values of the populations look to be over 2000 for the rabbits and less than 1500 for the foxes.This indicates even though the predation has e ect on the rabbits population,but their population starting with 50 continues to grow.

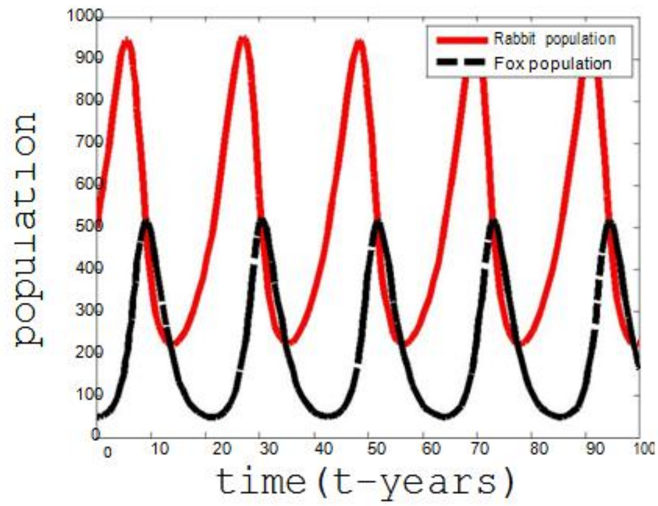

Figure 2: populations of the two species over 100 year span starting with R = 500,F = 50

In the second case,we interchanged the populations,that is,starting with a rabbit popula-tion of 500 and a fox population of 50.From gure 2,it also shows that the trend exhibits a periodic population density where a decline in rabbit population correlates also with decline in fox population. The cycle occurs in roughly 10 years increment.It also shows that an initially high population of rabbits reaching a maximum of over 900 declines as the fox population increases to maximum population of 500.The rabbits population decreases to the lowest popu-lation of roughly 200 while the foxes maintain its lowest same as its initial starting population of 50.The highest population attained by the foxes and the lowest population of the rabbits gives two points of intersection between the two populations.

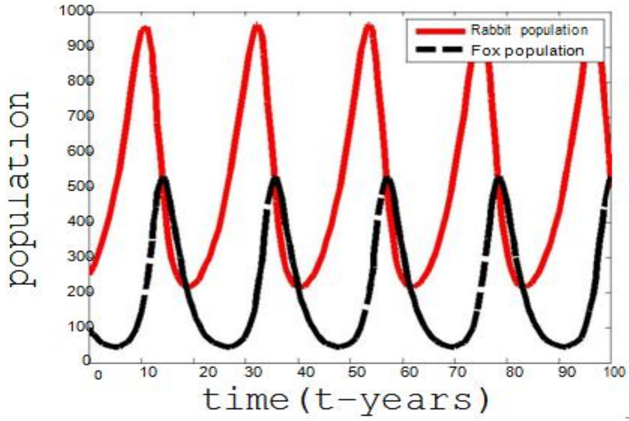

Figure 3: populations of the two species over 100 year span starting with R = 250,F = 100

In the third case, we started with an initial rabbit population of 250 and a fox population of 100 . The gure 3 shows the same trend to that of gure 2 indicating no signi cant change in the behavior of the model with this change of two populations.

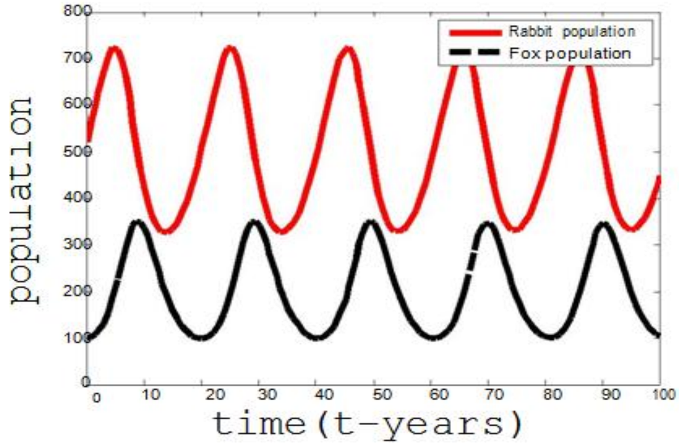

Figure 4: populations of the two species over 100 year span starting with R = 520,F = 100

In the fourth case,we changed an initial rabbit population to 520 and maintain a fox popu-lation of 100 as in the above.Figure 4 really shows that the trend di ers from the three previous but still follows a periodic trend. This trend provides more stable population than the previous trends and the values stay within a particular ranges of the two populations hinting no extreme rise and fall scenarios. The populations do not intersect each other though the relatively stable population lends some knowledge to what may be a numerical relationship between the two populations.

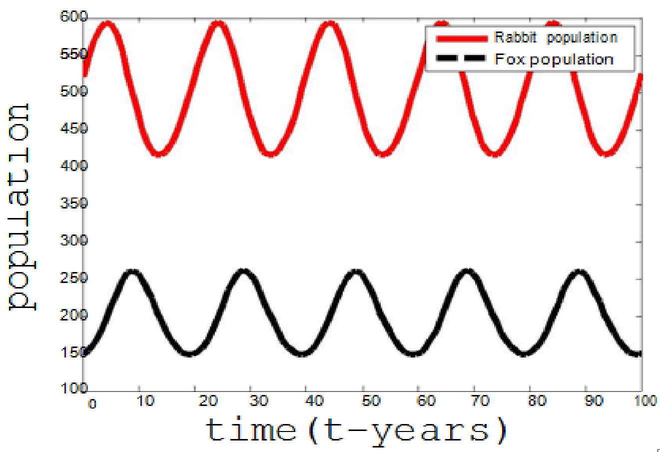

Figure 5: populations of the two species over 100 year span starting with R = 520,F = 150

In the fth case, we maintain an initial rabbit population of 250 and increased that of the fox to 150. Figure 5 shows that the trend is closely similar to that of Figure 4 in terms of corresponding values for range and the periodicity.

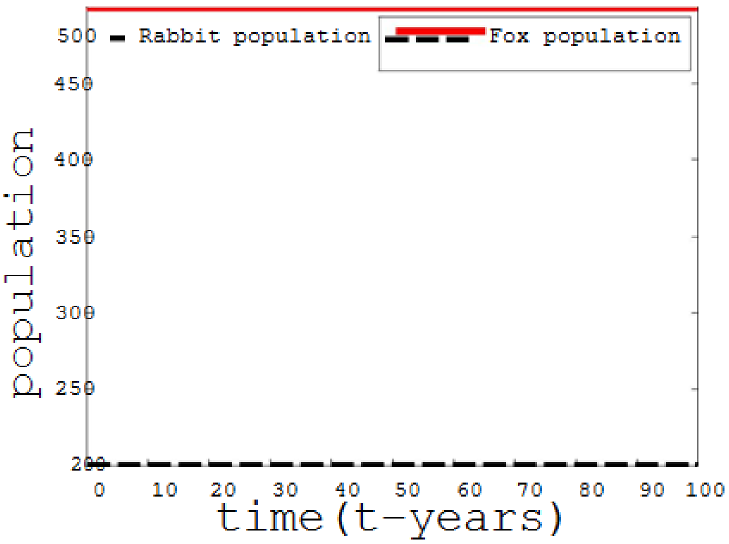

Figure 6: populations of the two species over 100 year span starting with R = 500,F = 200

In the last case, we choose initial populations of 500 for the rabbit and 200 for the fox.Figure 6 provides quite a di erent trend,thus providing a fully stable populations that are straight lines at those particular values. This shows that htere is no change in the two populations over 100 year time span. This choice also signi es that using the parameter values used in this paper,R = 500 and F = 200 will satisfy the equilibrium condition of the model.

V. Conclusion And Recommendation

We have so far investigated the behavior of the simple predator-prey model involving the inter-action between the rabbits(prey) and the foxes(predator) by exploring the trends visible from six projections of the two populations. After all to capture the essential dynamical features of the model and the scenarios,using the ODE45 solver in the MATLAB,certain important trends can be observed. From the gures,it supports the notion that the rabbit population and the fox population are codependent on one another which coincides with the usual observation from the eld between predator and prey in an ecosystem as reported in other literatures.Also the model shows a stable limit cycle in the positive octant,hence shows no extinction of the two populations and this plays a great role in the survival of the ecosystem in terms of game reserve and animal hunt in the forest.

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