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Reformulated Harary Indices of Composite Graphs

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Abstract
In this paper, we study a new graph invariant named reformulated Harary index $\overline{H}_t$, which is defined for a connected graph G as $\overline{H}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{1}{d_G(u,v)+t}$, $t \geq 0$. On the other hand, it is the generalized version of the Harary index. In this paper, we obtain the exact formulae for the reformulated Harary indices of some composite graphs such as join, disjunction, strong product, composition and tensor product of two graphs.

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I. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. A *join* $G+H$ of two graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ is the graph on the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)\cup\{uv \mid u\in V(G),\ v\in V(H)\}$. Hence, the join of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs.

The *disjunction* $G\vee H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and (u_1,v_1) is adjacent with (u_2,v_2) whenever $u_1u_2\in E(G)$ or $v_1v_2\in E(H)$. The *strong product* of graphs G and H, denoted by $G\boxtimes H$, is the graph with vertex set $V(G)\times V(H)=\{(u,v) : u\in V(G), v\in V(H)\}$ and $(u,x)(v,y)$ is an edge whenever (i) $u=v$ and $xy\in E(H)$, or (ii) $uv\in E(G)$ and $x=y$, or (iii) $uv\in E(G)$ and $xy\in E(H)$. The *composition* of the graphs G and H, denoted by $G\circ H$, has vertex set $V(G)\times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever g_1g_2 is an edge in G or, $g_1=g_2$ and h_1h_2 is an edge in H. For two simple graphs G and H their *tensor product,* denoted by $G\times H$, has vertex set $V(G)\times V(H)$ in which (g_1,h_1) and (g_2,h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G\times H$ is connected only if at least one of the graph is nonbipartite.

A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds [6]. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [10].

Let G be a connected graph. Then *Wiener index* of G is defined as $W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the *Harary index* of G is defined as $H(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{1}{d_G(u,v)}$.

The *reformulated Harary index* of a connected graph G is defined as $\overline{H}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{1}{d_G(u,v)+t}$, $t \geq 0$. The reformulated Harary index is the generalized version of the Harary index. It is natural and interesting to study the mathematical properties of this novel graph index. The Harary index of a graph G has been introduced independently by Plavsic et al. [8] and by Ivanciuc et al. [7] in 1993. Its applications and mathematical properties are well studied in [1, 2, 3, 9]. Zhou et al. [4] have obtained the lower and upper bounds of the Harary index of a connected graph. Very recently, Xu et al. [5] have obtained lower and upper bounds for the Harary index of a connected graph in relation to $\chi(G)$, chromatic number of G and $\omega(G)$, clique number of G. and characterized the extremal graphs that attain the lower and upper bounds of Harary index. Also, Feng et. al. [2] have given a sharp upper bound for the Harary indices of graphs based on the matching number, that is, the size of a maximum matching. In this paper, we obtain the exact formulae for the reformulated Harary indices of some composite graphs such as join, disjunction, strong product, composition and tensor product of two graphs

II. Composite Graphs

In this section, we compute the reformulated Harary indices of join, disjunction, strong product and composition of two connected graphs.

2.1 Join

First we obtain the reformulated Harary indices of join of two graphs.

Theorem 2.1. *Let G_1 and G_2 be graphs with n, m vertices and p, q edges, respectively. Then* $\overline{H}_t(G_1+G_2) = \frac{p+q+mn}{1+t} + \frac{1}{2+t}\Big(\frac{n(n-1)+m(m-1)}{2} - (p+q)\Big)$, $t \geq 0$.

Proof. By the structure of the join of two graphs, we have

$$d_{G+H}(u,v) = \begin{cases} 0, \text{ if } u = v \\ 1, \text{ if } uv \in E(G) \text{ or } uv \in E(H) \text{ or } (u \in V(G) \text{ and } v \in V(H)) \\ 2, \text{ otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
\overline{H}_t(G_1+G_2) &= \frac{1}{2}\sum_{u,v\in V(G_1+G_2)} \frac{1}{d_{G_1+G_2}(u,v)+t} \\
&= \frac{1}{2}\Bigg(\sum_{uv\in E(G_1)} \frac{1}{d_{G_1+G_2}(u,v)+t} + \sum_{uv\notin E(G_1)} \frac{1}{d_{G_1+G_2}(u,v)+t} + \sum_{uv\in E(G_2)} \frac{1}{d_{G+H}(u,v)+t} \\
&\quad + \sum_{uv\in E(G_2)} \frac{1}{d_{G_1+G_2}(u,v)+t} + \sum_{u\in V(G_1),\, v\in V(G_2)} \frac{1}{d_{G_1+G_2}(u,v)+t}\Bigg) \\
&= \frac{p}{1+t} + \frac{1}{2+t}\Big(\frac{n(n-1)}{2} - p\Big) + \frac{q}{1+t} + \frac{1}{2+t}\Big(\frac{m(m-1)}{2} - q\Big) + \frac{mn}{1+t} \\
&= \frac{p+q+mn}{1+t} + \frac{1}{2+t}\Big(\frac{n(n-1)+m(m-1)}{2} - (p+q)\Big). \qquad \blacksquare
\end{aligned}$$

If we consider $t = 0$, in Theorem 2.1, we obtain the Harary index of join of two graphs.

Corollary 2.2. *Let G_1 and G_2 be graphs with n, m vertices and p, q edges, respectively. Then* $H(G_1+G_2) = mn + \frac{1}{2}(p+q) + \frac{1}{4}(n(n-1) + m(m-1))$. ■

Using Corollary 2.2, we have the following corollary.

Corollary 2.3. *Let G be graph on n vertices. Then* $H(G+K_m) = mn + \frac{1}{2}|E(G)| + \frac{1}{4}\big(n(n-1) + m(m-1)\big)$. ■

Using Corollary 2.3, we compute the formula for Harary indices of fan and wheel graphs, $P_n + K_1$ and $C_n + K_1$.

Example 2.4. *(i)* $H(P_n + K_1) = \frac{1}{4}(n^2 + 5n - 2)$.
(ii) $H(C_n + K_1) = \frac{1}{4}(n^2 + 5n)$.

2.2 Disjunction

Here we obatin the reformulated Harary indices of disjunction of two graphs.

Theorem 2.5. *Let G_1 and G_2 be graphs with n, m vertices and p, q edges, respectively. Then* $\overline{H}_t(G_1 \vee G_2) = \frac{1}{1+t}\big((m^2 - m)p + (n^2 + n)q\big) + \frac{1}{2+t}\big(\frac{m^2n^2+mn-1}{2} - m^2p - n^2q\big)$.

Proof. Set $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G_1 \vee G_2$. From the structure of the disjunction of two graphs, we have

$$d_{G\vee H}((u_i, v_j), (u_k, v_r)) = \begin{cases} 0, \text{ if } u_i = u_k \text{ and } v_j = v_r \\ 1, \text{ if } u_iu_k \in E(G) \text{ or } v_jv_r \in E(H) \\ 2, \text{ otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \overline{H}_t(G_1 \vee G_2) &= \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G_1 \vee G_2)} \frac{1}{d_{G_1\vee G_2}(x_{ij}, x_{kp}) + t} \\ &= \frac{1}{2} \sum_{x_{ij} \in V(G_1\vee G_2)} \Big\{\frac{1}{1+t}\big(md(u_i) + nd(v_j) - d(u_i) + d(v_j)\big) \\ &\quad + \frac{1}{2+t}\big(mn - md(u_i) - nd(v_j) + d(u_i)d(v_j) - 1\big)\Big\} \\ &= \frac{1}{2(1+t)}\big(2m^2p + 2n^2q - 2mp + 2nq\big) + \frac{1}{2(2+t)}\big(m^2n^2 - 2m^2p - 2n^2q + mn - 1\big) \\ &= \frac{1}{1+t}\big((m^2 - m)p + (n^2 + n)q\big) + \frac{1}{2+t}\big(\frac{m^2n^2 + mn - 1}{2} - m^2p - n^2q\big). \blacksquare \end{aligned}$$

If we consider $t = 0$, in above theorem, we have the following corollary.

Corollary 2.6. *Let G_1 and G_2 be graphs with n, m vertices and p, q edges, respectively. Then* $H(G_1 \vee G_2) = \frac{m^2}{2}p + \frac{m^2}{2}q - pq + \frac{1}{4}mn(mn - 1)$. ■

2.3 Strong product

Here we obtain the reformulated Harary indices of strong product of two graphs.

Theorem 2.7. *Let G be a connected graph with n vertices and m edges. Then* $\overline{H}_t(G \boxtimes K_r) = r^2\overline{H}_t(G) + \frac{1}{2(1+t)}nr(r - 1)$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \boxtimes K_r$. One can see that for any pair of vertices x_{ij}, $x_{kp} \in V(G \boxtimes K_r)$, $d_{G\boxtimes K_r}(x_{ij}, x_{ip}) = 1$ and $d_{G\boxtimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

$$\begin{aligned} \overline{H}_t(G \boxtimes K_r) &= \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G\boxtimes K_r)} \frac{1}{d_{G\boxtimes K_r}(x_{ij}, x_{kp}) + t} \\ &= \frac{1}{2}\Big(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{d_{G\boxtimes K_r}(x_{ij}, x_{ip}) + t} + \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G\boxtimes K_r}(x_{ij}, x_{kj}) + t} \\ &\quad + \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{d_{G\boxtimes K_r}(x_{ij}, x_{kp}) + t}\Big) \\ &= \frac{1}{2}\Big(\frac{nr(r-1)}{1+t} + 2r\overline{H}_t(G) + 2r(r-1)\overline{H}_t(G)\Big) \\ &= r^2\overline{H}_t(G) + \frac{1}{2(1+t)}nr(r-1). \end{aligned}$$

■

If we consider $t = 0$, in above theorem we have the following corollary.

Corollary 2.8. *Let G be a connected graph with n vertices and m edges. Then $H(G \boxtimes K_r) = r^2 H(G) + \frac{1}{2}nr(r-1)$.* ■

By direct calculations we obtain expressions for the values of the Harary indices of P_n and C_n.

$$H(P_n) = n\Big(\sum_{i=1}^{n} \frac{1}{i}\Big) - n \text{ and } H(C_n) = \begin{cases} n\Big(\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 1 & \text{n is even} \\ n\Big(\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big) & \text{n is odd.} \end{cases}$$

As an application we present formulae for Harary indices of open and closed fences, $P_n \boxtimes K_2$ and $C_n \boxtimes K_2$.

By using corollary 2.8, $H(C_n)$ and $H(P_n)$, we obtain the exact Harary indices of the following graphs.

Example 2.9. (*i*) $H(P_n \boxtimes K_2) = n\Big(4\sum_{i=1}^{n} \frac{1}{i} - 3\Big)$.

$$(ii) \quad H(C_n \boxtimes K_2) = \begin{cases} n\Big(1 + 4\sum_{i=1}^{\frac{n}{2}} \frac{1}{i}\Big) - 4 & n \text{ is even} \\ n\Big(1 + 4\sum_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\Big) & n \text{ is odd.} \end{cases}$$

III. Composition

In this section, we obtain the reformulated Harary index of $G_1 \circ G_2$.

Theorem 3.1. *Let G_1 and G_2 be two connected graphs with n, m vertices and p, q edges, respectively. Then $\overline{H}_t(G_1 \circ G_2) = m^2 \overline{H}_t(G_1) + \frac{n}{2}\Big(\frac{2q}{1+t} + \frac{(m^2-2q-m)}{2+t}\Big)$.*

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and let $V(G_2) = \{v_1, v_2, \ldots, v_m\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G_1 \circ G_2$. By the definition of t-Harary index

$$\begin{aligned} \overline{H}_t(G_1 \circ G_2) &= \frac{1}{2} \sum_{x_{ij}, x_{k\ell} \in V(G_1 \circ G_2} \frac{1}{d_{G_1 \circ G_2}(x_{ij}, x_{k\ell}) + t} \\ &= \frac{1}{2}\Big(\sum_{i=0}^{n-1} \sum_{\substack{j,\ell=0 \\ j \neq \ell}}^{m-1} \frac{1}{d_{G_1 \circ G_2}(x_{ij}, x_{i\ell}) + t} + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{j=0}^{m-1} \frac{1}{d_{G_1 \circ G_2}(x_{ij}, x_{kj}) + t} \\ &\quad + \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \sum_{\substack{j,\ell=0 \\ j \neq \ell}}^{m-1} \frac{1}{d_{G_1 \circ G_2}(x_{ij}, x_{k\ell}) + t} \Big). \end{aligned}$$

We shall calculate the above sums are separately.

$$\sum_{i=0}^{n-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{m-1}\frac{1}{d_{G_1\circ G_2}(x_{ij},x_{i\ell})+t} = n\left(\sum_{v_jv_\ell\in E(G_2)}\frac{1}{d_{G_2}(v_j,v_\ell)+t}+\sum_{v_jv_\ell\notin E(G_2)}\frac{1}{d_{G_2}(v_j,v_\ell)+t}\right)$$
$$= n\left(\sum_{v_j\in V(G_2)}\frac{deg(v_j)}{1+t}+\sum_{v_j\in V(G_2)}\frac{(m-deg(v_j)-1)}{2+t}\right).$$

Since each row induces a copy of G_2 and $d_{G_1\circ G_2}(x_{ij},x_{i\ell}) = \begin{cases}1, & \text{if } v_jv_\ell\in E(G_2)\\ 2, & \text{if } v_jv_\ell\notin E(G_2).\end{cases}$

$$\sum_{i=0}^{n-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{m-1}\frac{1}{d_{G_1\circ G_2}(x_{ij},x_{i\ell})+t} = n\left(\frac{2q}{1+t}+\frac{(m^2-2q-m)}{2+t}\right). \quad (3.1)$$

$$\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{m-1}\frac{1}{d_{G_1\circ G_2}(x_{ij},x_{kj})+t} = m\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\frac{1}{d_{G_1}(u_i,u_k)+t}.$$

Since the distance between a pair of vertices in a column is same as the distance between the corresponding vertices of other column.

$$\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{m-1}\frac{1}{d_{G_1\circ G_2}(x_{ij},x_{kj})+t} = 2m\overline{H}_t(G_1). \quad (3.2)$$

$$\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{m-1}\frac{1}{d_{G_1\circ G_2}(x_{ij},x_{k\ell})+t} = m(m-1)\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\frac{1}{d_{G_1}(u_i,u_k)+t}$$
$$= 2m(m-1)\overline{H}_t(G_1). \quad (3.3)$$

By (3.1), (3.2) and (3.3), we have,

$\overline{H}_t(G_1\circ G_2) = m^2\overline{H}_t(G_1)+\frac{n}{2}\left(\frac{2q}{1+t}+\frac{(m^2-2q-m)}{2+t}\right).$ ■

If we consider $t=0$, in above theorem we have the following corollary.

Corollary 3.2. *Let G_1 and G_2 be two connected graphs with n, m vertices and p, q edges, respectively. Then $H(G_1\circ G_2) = m^2H(G_1)+\frac{n}{4}\left(m^2+2q-m\right)$.* ■

3.1 Tensor product

The proof of the following lemma follows easily from the properties and structure of $G \times K_r$.

Lemma 3.3. *Let G be a connected graph on $n \geq 2$ vertices. For any pair of vertices $x_{ij}, x_{kp} \in V(G \times K_r)$, $r \geq 3$, $i,k \in \{1,2,\ldots,n\}$ $j,p \in \{1,2,\ldots,r\}$. Then*

(i) If $u_iu_k \in E(G)$, then

$$d_{G\times K_r}(x_{ij}, x_{kp}) = \begin{cases} 1, & \text{if } j \neq p, \\ 2, & \text{if } j = p \text{ and } u_iu_k \text{ is on a triangle of } G, \\ 3, & \text{if } j = p \text{ and } u_iu_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) If $u_iu_k \notin E(G)$, then $d_{G\times K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

(iii) $d_{G\times K_r}(x_{ij}, x_{ip}) = 2$. ■

Now we compute the reformulated Harary index of $G \times K_r$.

Theorem 3.4. *Let G be a connected graph with $n \geq 2$ vertices and m edges and let μ be the number of edges of G which do not lie on any C_3 of it. Then $\overline{H}_t(G \times K_r) = r^2\overline{H}_t(G) + \frac{nr(r-1)}{2(2+t)} - \frac{2rm}{(1+t)(2+t)} - \frac{2r\mu}{(2+t)(3+t)}$, where $r \geq 3$.*

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. From the definition of $\overline{H}_t$

$$\begin{aligned} \overline{H}_t(G \times K_r) &= \frac{1}{2} \sum_{x_{ij}, x_{kp} \in V(G\times K_r)} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kp}) + t} \\ &= \frac{1}{2}\Bigg(\sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{ip}) + t} + \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{j=0}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj}) + t} \\ &\quad + \sum_{\substack{i,k=0 \\ i\neq k}}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kp}) + t} \Bigg). \end{aligned}$$

First we compute $S_1 = \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{ip}) + t}$.

$$\begin{aligned} S_1 &= \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{ip}) + t} \\ &= \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} \frac{1}{2+t}, \text{ by Lemma 3.3} \\ &= \frac{nr(r-1)}{2+t}. \end{aligned}$$

Next we compute $S_2 = \sum_{j=0}^{r-1} \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj})+t}$.

Let $E_1 = \{uv \in E(G) \mid uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$.

$$\begin{aligned}
S_2' &= \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj})+t} \\
&= \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \notin E(G)}}^{n-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj})+t} + \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_1}}^{n-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj})+t} + \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_2}}^{n-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj})+t} \\
&= \left(\sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \notin E(G)}}^{n-1} \frac{1}{d_G(u_i, u_k)+t} + \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_1}}^{n-1} \frac{1}{2+t} + \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_2}}^{n-1} \frac{1}{3+t} \right), \text{ by Lemma 3.3}
\end{aligned}$$

Add and subtract $\frac{1}{1+t}$ on both second and third sums, we have

$$\begin{aligned}
S_2' &= \left(\sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \notin E(G)}}^{n-1} \frac{1}{d_G(u_i, u_k)+t} + \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_1}}^{n-1} \frac{1}{d_G(u_i, u_k)+t} + \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_2}}^{n-1} \frac{1}{d_G(u_i, u_k)+t} \right) \\
&\quad - \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_1}}^{n-1} \frac{1}{(1+t)(2+t)} - \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_2}}^{n-1} \frac{2}{(1+t)(3+t)}, \text{ since } d_G(u_i, u_k) = 1, \text{ in the second and third sums}
\end{aligned}$$

Add and subtract $\frac{1}{(1+t)(2+t)}$ on the third sums, we have

$$\begin{aligned}
S_2' &= \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \frac{1}{d_G(u_i, u_k)+t} - \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_1}}^{n-1} \frac{1}{(1+t)(2+t)} - \sum_{\substack{i,k=0 \\ i \neq k \\ u_iu_k \in E_2}}^{n-1} \frac{1}{(2+t)(3+t)} \\
&= 2\overline{H}_t(G) - \frac{2m}{(1+t)(2+t)} - \frac{2\mu}{(2+t)(3+t)}, \text{ where } m \text{ and } \mu \text{ are the numbers of edges of } G
\end{aligned}$$

which do not on any C_3 and edges of G, respectively. (3.4)

Now summing (3.4) over $j = 0, 1, \ldots, r-1$, we get,

$$\begin{aligned}
S_2 &= \sum_{j=0}^{r-1} \left(\sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kj})+t} \right) \\
&= 2r\overline{H}_t(G) - \frac{2mr}{(1+t)(2+t)} - \frac{2r\mu}{(2+t)(3+t)}r.
\end{aligned}$$

Finally, we compute $S_3 = \sum_{\substack{i,k=0 \\ i \neq k}}^{n-1} \left(\sum_{\substack{j,p=0 \\ j \neq p}}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij}, x_{kp})+t} \right)$.

$$\begin{aligned} S_3 &= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \sum_{\substack{j,p=0,\\ j\neq p}}^{r-1} \frac{1}{d_{G\times K_r}(x_{ij},x_{kp})+t} \\ &= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \sum_{\substack{j,p=0,\\ j\neq p}}^{r-1} \frac{1}{d_G(u_i,u_k)+t}, \text{ by Lemma 3.3} \\ &= r(r-1)\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \frac{1}{d_G(u_i,u_k)+t} \\ &= 2r(r-1)\overline{H}_t(G). \end{aligned}$$

Add S_1 to S_3 and divide $\frac{1}{2}$, we arrive the desired result. ■

Using $t = 0$ in the above theorem, we obtain the following corollary.

Corollary 3.5. *Let G be a connected graph with $n \geq 2$ vertices and m edges and let μ be the number of edges of G which do not lie on any C_3 of it. Then $H(G\times K_r) = r^2H(G)+\frac{nr(r-1)}{4}-\frac{rm}{2}-\frac{r\mu}{3}$, where $r \geq 3$.* ■

Using above corollary, we obtain the following corollaries.

Corollary 3.6. *Let G be a connected graph on $n \geq 2$ vertices with m edges. If each edge of G is on a C_3, then $H(G\times K_r) = r^2\,H(G) + \frac{nr(r-1)}{4} - \frac{mr}{2}$, where $r \geq 3$.* ■

Corollary 3.7. *If G is a connected triangle free graph on $n \geq 2$ vertices and m edges, then $H(G\times K_r) = r^2\,H(G) + \frac{nr(r-1)}{4} - \frac{2mr}{3}$, where $r \geq 3$.* ■

By using Corollary 3.7, $H(P_n)$ and $H(C_n)$, we obtain the exact Harary indices of the following graphs.

Example 3.8. *(i) If $n \geq 2$ and $r \geq 3$ then $H(P_n\times K_r) = nr^2\Big(\sum_{i=1}^{n}\frac{1}{i}\Big) - \frac{r}{12}(11n+9rn-8)$.*

(ii) $$H(C_n\times K_r) = \begin{cases} r^2\Big\{n\Big(\sum_{i=1}^{\frac{n}{2}}\frac{1}{i}\Big)-1\Big\} + \frac{nr}{12}(3r-11), & \text{if } n \text{ is even} \\ \frac{3r(5r-3)}{4}, & \text{if } n=3 \\ r^2 n\Big(\sum_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\Big) + \frac{nr}{12}(3r-11), & \text{if } n>3 \text{ is odd.} \end{cases}$$

3.2 More operations

Let G be a connected graph. If we put two similar graphs G side by side, and any vertex of the first graph G is connected by edges with those vertices which are adjacent to the corresponding vertex of the second graph G and the resultant graph is denoted by $K_2 \bullet G$, then we have $V(K_2 \bullet G) = 2V(G)$ and $E(K_2 \bullet G) = 4E(G)$. Moreover, $K_2 \bullet G$ is the graph of K_2 and G with the vertex set $V(K_2 \bullet G) = V(K_2)\times V(G)$ and $(u_i, v_j)(u_k, v_r)$ is an edge of $K_2 \bullet G$ whenever $u_i = u_k$ and $v_jv_k \in E(G)$ or $u_i \neq u_k$ and $v_jv_k \in E(G)$.

Theorem 3.9. *Let G be a connected graph with n vertices. Then* $\overline{H}_t(K_2 \bullet G) = 4\overline{H}_t(G) + \frac{n}{2+t}$.

Proof. By the definition of the reformulated Harary index, we have

$$\begin{aligned}
\overline{H}_t(K_2 \bullet G) &= \sum_{(u_i,v_j),(u_k,v_r)\in V(K_2\bullet G),(u_i,v_j)\neq(u_k,v_r)} \frac{1}{d_{K_2\bullet G}((u_i,v_j),(u_k,v_r))+t} \\
&= \sum_{(u_i,v_j),(u_i,v_r)\in V(K_2\bullet G),j\neq r} \frac{1}{d_{K_2\bullet G}((u_i,v_j),(u_i,v_r))+t} \\
&\quad + \sum_{(u_i,v_j),(u_k,v_r)\in V(K_2\bullet G),i\neq k} \frac{1}{d_{K_2\bullet G}((u_i,v_j),(u_k,v_r))+t} \\
&= \sum_{u_i\in V(K_2)} \sum_{v_jv_k\in V(G),j\neq r} \frac{1}{d_G(v_j,v_r)+t} + \sum_{v_j\in V(G)} \left(\frac{1}{2+t} + \sum_{v_r\in V(G),j\neq r} \frac{1}{d_G(v_j,v_r)+t}\right) \\
&= 2\sum_{v_jv_k\in V(G),j\neq r} \frac{1}{d_G(v_j,v_r)+t} + \sum_{v_j\in V(G)} \sum_{v_r\in V(G),j\neq r} \frac{1}{d_G(v_j,v_r)+t} + \frac{n}{2+t} \\
&= 4\overline{H}_t(G) + \frac{n}{2+t}. \blacksquare
\end{aligned}$$

Using $t = 0$ in the above theorem, we obtain the following corollary.

Corollary 3.10. *Let G be a connected graph with n vertices. Then* $H(K_2 \bullet G) = 4H(G) + \frac{n}{2}$. ■

Let G be a connected graph. If we put two similar graphs G side by side, and any vertex of the first graph G is connected by edges with those vertices which are nonadjacent to the corresponding vertex (including the corresponding vertex itself) of the second graph G and the resultant graph is denoted by $K_2 \star G$, then we have $V(K_2 \star G) = 2V(G)$ and $E(K_2 \star G) = |V(G)|^2$. Moreover, $K_2 \star G$ is the graph of K_2 and G with the vertex set $V(K_2 \star G) = V(K_2) \times V(G)$ and $(u_i, v_j)(u_k, v_r)$ is an edge of $K_2 \star G$ whenever $u_i = u_k$ and $v_jv_k \in E(G)$ or $u_i \neq u_k$ and $v_jv_k \notin E(G)$.

Theorem 3.11. *Let G be a connected graph with n vertices. Then* $\overline{H}_t(K_2 \star G) = \frac{n(t+n(t+3)-1)}{(1+t)(2+t)}$.

Proof. By the definition of the Wiener index, we have

$$\begin{aligned}
\overline{H}_t(K_2 \star G) &= \sum_{(u_i,v_j),(u_k,v_r)\in V(K_2\star G),(u_i,v_j)\neq(u_k,v_r)} \frac{1}{d_{K_2\star G}((u_i,v_j),(u_k,v_r))+t} \\
&= \frac{1}{2}\sum_{u_i\in K_2}\sum_{v_j\in V(G)} \Big(\frac{d_{K_2\star G}(u_i,v_j)}{1+t} + \frac{1}{2+t}(2n - d_{K_2\star G}(u_i,v_j) - 1)\Big),
\end{aligned}$$

since for every vertex $(u_i, v_j) \in V(K_2 \star G)$, there are $d_G(v_j) + n - 1 - d_G(v_j) + 1 = n$ neighbors, and $d_G(v_j) + n - 1 - d_G(v_j) = n - 1$ vertices with the distance 2 from itself

$$\begin{aligned}
&= \frac{1}{2}\sum_{u_i\in K_2}\sum_{v_j\in V(G)} \Big(\frac{n}{1+t} + \frac{n-1}{2+t}\Big) \\
&= \frac{n(t+n(t+3)-1)}{(1+t)(2+t)}. \blacksquare
\end{aligned}$$

Using $t = 0$ in the above theorem, we obtain the following corollary.

Corollary 3.12. *Let G be a connected graph with n vertices. Then* $H(K_2 \star G) = \frac{n(3n-1)}{2}$. ■

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