IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 13, Issue 5 Ver. III (Sep. - Oct. 2017), PP 67-72
www.iosrjournals.org

Boosting Invariants To Equivariants For Coupled Takens-Bogdanov Systems

Grace Gachigua[1], David Malonza[2]
[1]Department of Pure and Applied Mathematics The Technical University of Kenya P.O. Box 52428, 00200, Nairobi, Kenya.
[2]Department of Mathematics Kenyatta University P.O. Box 43844 , 00200, Nairobi, Kenya.
Corresponding Author Email: wambuigachigua@gmail.com

Abstract: *When a set of vector fields is in normal forms this helps to understand the behavior of general nonlinear systems. In this work we describe the normal form of formal power series vector fields on* R^{2n} *whose linear terms are the nilpotent matrix made up of n* 2×2 *Jordan blocks. We use an algorithm based on the notion of transvectants from classical invariant theory in determining the normal form for the system with nilpotent linear part.*

Keywords: *transvectant, equivariants, box product, Stanley decomposition*

Date of Submission: 04-10-2017 Date of acceptance: 28-10-2017

I. Introduction

The term normal form is widely used in mathematics and its meaning is very sensitive for the context. In the case of linear maps from a given vector space to itself, for example, one may consider all possible choices of a basis. Each choice gives a matrix-representation of a given linear map. A suitable choice of basis now gives the well-known Jordan canonical form. This normal form, in a simple way displays certain important properties of the linear map, concerning its spectrum, its eigenspaces, and so on.

There are well-known procedures for putting a system of differential equations $\dot{x}=Ax+v(x)$(where v is a formal power series with quadratic terms) into normal form with respect to its linear part A. Our concern in this paper is to describe the normal form of space A, that is the set of all v such that $Ax+v(x)$ is in normal form. Our main result is a procedure that solves the description problem where N, the nilpotent matrix in Jordan form, with coupled n Jordan blocks, provided that the description problem is already solved for each Jordan block of N taken separately. Our method is based on adding one block at a time. This procedure will be illustrated with examples. Our goal is to compute the algorithm for determining normal form for linear system with linear part $N_{33\ldots3}$ from the Stanley decomposition of the ring of invariants.

The idea of simplification near an equilibrium goes back at least to Poincare in the year 1880, who was among the first to bring forth the theory in a more definite form. Poincare considered the problem of reducing a system of nonlinear differential equations to a system of linear ones. The formal solution of this problem entails finding near-identity coordinate transformations, which eliminate the analytic expressions of the nonlinear terms.

rom [2], Cushman et al, using a method called Co-variant of special equivariant solved the problem of finding Stanley decomposition of $N_{22,\ldots,2}$. Their method begins by creating a scalar problem that is larger than the vector problem and their procedures are derived from classical invariant theory thus it was necessary to repeat calculations of classical theory at the levels of equivariants.

In [8] Namachchivaya et al, studied a generalized Hopf bifurcation with non-semisimple 1:1 Resonance. The normal form for such a system contains only terms that belong to both the semisimple part of A and the normal form of the nilpotent, which is a couple Takens-Bogdanov system with $A=\begin{bmatrix} i\omega & 1 & & \\ & i\omega & & \\ & & i\omega & 1 \\ & & & i\omega \end{bmatrix}$.

This example illustrates the physical significance of the study of normal forms for systems with nilpotent linear part.

Murdock [6] developed an efficient algorithm for the production of Stanley decomposition of the module of the equivariants from the ring of invariants. The nonlinear terms are decomposed into essential and non essential terms and thus the normal form of nonlinear system can be computed. In 2004, Malonza [4], used the proposed technique in [6] for Sl(2) (inner product) for finding the normal form of $N_{222...}$ by computing Stanley decomposition for equivariants of nilpotent systems consisting of Jordan blocks.

In [7], Murdock and Sanders offered an algorithm to determine the form of normal form or a vector field with nilpotent linear part, when the form of the normal form is known for each Jordan block of the linear part taken separately. The algorithm is based on the notion of transvectants from the classical invariant theory. Our results are mainly based on the work found in [7,8] that is application of transvectant's method for computing normal form for the ring of equivariants of nilpotent systems. In section two and three we together background knowledge on understanding the content of this work. In section four forms the central part of this paper where we shall compute the rings of equivariants.

II. Describing invariant rings by Stanley Decompositions

For any nilpotent matrix N, we define the Lie operator

$$L_N:P_j(R^n,R^n)\rightarrow P_j(R^n,R^n) \quad (1)$$

by

$$(L_N v)x=v'(x)Nx-Nv(x) \quad (2)$$

and the differential operator

$$D_{Nx}:P_j(R^n,R)\rightarrow P_j(R^n,R) \quad (3)$$

by

$$(D_{Nx}f)(x)=f'(x)Nx=(Nx.\triangledown)f(x). \quad (4)$$

A function f is called an invariant of Ax if $\frac{\partial}{\partial t}f(e^{At}x)|_{t=0}=0$ or equivalently $D_A f=0$ or $f\in kerD_A$. Similarly a vector field v is called a *equivariants* of Ax, if $\frac{\partial}{\partial t}(e^{-At}v(e^{At}x))|_{t=0}=0$ that is $L_A vx=0$ or $v\in kerL_A$.

There are two normal form styles in common use for nilpotent systems, the *inner product normal form* and the *sl(2) normal form.* The inner product normal form is defined by $P(R^n,R^n)=imL_N\oplus kerL_{N^*}$ where N^* is the conjugate transpose of N, a nilpotent matrix. To define the sl(2) normal form, one first sets $X=N$ and constructs matrices Y and Z such that

$$[X,Y]=Z, \quad [Z,X]=2X, \quad [Z,Y]=-2Y. \quad (5)$$

Having obtained the triad $\{X,Y,Z\}$ we create two additional triads $\{X,Y,Z\}$ and $\{\mathsf{X},\mathsf{Y},\mathsf{Z}\}$ as follows

$$X=D_Y, \quad Y=D_X, \quad Z=D_Z \quad (6)$$

$$(7)$$

Observe that the operators $\{\mathsf{X},\mathsf{Y},\mathsf{Z}\}$ map each $P(R^n,R^n$ into itself. It then follows from the representation theory $sl(2)$ that

$$P(R^n,R^n)=im\mathsf{Y}\oplus ker\mathsf{X}=im\mathsf{X}\oplus ker\mathsf{Y}$$

III. Boosting to rings of invariants to rings of equivariants

In this section we describe the procedure for obtaining a Stanley decomposition of the module of equivariants (or normal form space $kerX$) from a Stanley decomposition of the ring of invariants $kerX$; here $X=L_{N^*}$, just as $X=D_{N^*}$.

The module of all formal power series vector fields on R^n can be viewed as the tensor product $R[[x_1,...,x_n]]\otimes R^n$, and in fact the tensor product can be identified with the ordinary product (of a field times a constant vector) since the ordinary product satisfies the same algebraic rules as a tensor product. Specifically, every formal power series vector field can be written as

$$f_1(x)e_1+\dots+f_n(x)e_n=\begin{bmatrix} f_1(x). \\ \cdot \quad \cdot \\ f_n(x) \end{bmatrix}$$

where the e_i are the standard basis vectors of R^n. Next, the Lie derivative $X=L_N*$ can be expressed as the tensor product of X and $-N^*$, that is $X=X\otimes I+I\otimes(-N^*)$. Under the identification of $\otimes$ with ordinary product, this means $X(fv)=(Xf)v+f(-N^*)$, where $f\in R[[x_1,\dots,x_n$ and $v\in R^n$ in agreement with the following calculation, in which $v'=0$ because v is constant.

$$\begin{aligned} X(fv) &= L_N*(fv) \\ &= (D_N*f)v+f(L_N*v) \\ &= (D_N*f)v+f(v'N^*x-N^*v) \\ &= (D_N*)v+f(-N^*v). \end{aligned}$$

This kind of calculation also shows that $sl(2)$ representation (on vector fields) with triad (X,Y,Z) is the tensor product of the representation (on scalar fields) with triad (X,Y,Z) and the representation (on R^* with triad $-N^*,-M^*,-H$).

It follows that a basis from the normal form space $kerX$ is given by the well defined transvectants $(f,v)^{(i)}$ as f ranges over a basis for $kerX\subset R[[x_1,\dots,x_n]]$ and v ranges over a basis for $kerN^*\subset R^n$. (Of course $ker-(N^*=kerN^*$). The first of these bases is given by the standard monomials of a Stanley decomposition for $kerX$. The second is given by the standard basis vectors $e_r\in R$ such that r is the index of the bottom row of a jordan block in N^* (or equivalently, in N). It is useful to note that the weight of such an e_r is one less than the size of the block. The definition of transvectant in this case becomes

$$(f,e_r)^i=\sum_{j=0}^{i}(-1)^j W^{i,j}_{f,e_r}(\mathbf{Y^j f})((\mathbf{-M^*})^{\mathbf{j-1}}e_r)$$

$$=(f,g)^i=\sum_{j=0}^{i}(-1)^j W^{i,j}_{f,g}(\mathbf{Y^j f})((\mathbf{M^*})^{\mathbf{j-1}}g).$$

The computational procedures are the same as those used in describing rings of invariants from [8], except that infinite iterations never arise.

IV. Normal Form for $N_{22,\dots,2}$ System

We shall consider the normal form for nonlinear system with linear part having two and three blocks.

4.1 *Normal Form for N_{22} System*

The Stanley decomposition for ring of invariants with linear part N_{22} is given by $R[[\alpha_1,\alpha_2,\beta_{1,2}]]$. Since $\beta_{1,2}$ has weight zero, it is convenient to suppress it since we do not expand along terms of weight zero by setting $R=[[\beta_{1,2}]]$ and writing

$$ker\,X_{22} = R[[\alpha_1,\alpha_2]]$$

Expanding the Stanley decomposition:

$$\begin{aligned} ker\, X_{22} &= R[[\alpha_2]]\oplus R[[\alpha_1,\alpha_2]]\alpha_1 \\ &= R\oplus R[[\alpha_2]]\alpha_2\oplus R[[\alpha_1,\alpha_2]]\alpha_1 \end{aligned}$$

Since we have two blocks the basis elements are e_2 and e_4 which are both of weight 1. Therefore we need to compute the box product of the ring $ker\, X_{22}$ with $Re_2\oplus Re_4$. That is

$\ker X_{22} = (ker\, X_{22})\boxtimes(Re_2\oplus Re_4)$.

Distributing the box product, there are two cases to consider.

Case 1: $[[R\oplus R[[\alpha_2]]\alpha_2\oplus R[[\alpha_1,\alpha_2]]\alpha_1]]\boxtimes Re_2$

$=Re_2\oplus R[[\alpha_2]]\alpha_2 e_2\oplus R[[\alpha_2]](\alpha_2,e_2)^{(1)}\oplus$

$R[[\alpha_1,\alpha_2]]\alpha_1 e_2\oplus R[[\alpha_1,\alpha_2]](\alpha_1,e_2)^{(1)}$

Case 2: Similarly, we have

$[R\oplus R[[\alpha_2]]\alpha_2\oplus R[[\alpha_1,\alpha_2]]\alpha_1]\boxtimes Re_4=$

$R[[\alpha_1,\alpha_2]]e_4\oplus R[[\alpha_2]](\alpha_2,e_4)^{(1)}\oplus R[[\alpha_1,\alpha_2]](\alpha_1,e_6)^{(1)}$

Adding the terms in cases 1 and 2 we obtain:

$$\begin{aligned} \ker X_{22} &= R[[\alpha_1,\alpha_2]]e_2\oplus R[[\alpha_1,\alpha_2]](\alpha_1,e_2)^{(1)}\oplus \\ &\quad R[[\alpha_2]](\alpha_2,e_2)^{(1)}\oplus R[[\alpha_1,\alpha_2]]e_4\oplus \\ &\quad R[[\alpha_1,\alpha_2]](\alpha_1,e_4)^{(1)}\oplus R[[\alpha_2]](\alpha_2,e_4)^{(1)} \quad (8) \end{aligned}$$

To complete the calculation, it is necessary to compute the transvectants that appear from the cases of the normal form.. These are $(f,e_2)^{(i)}$ and $(f,e_4)^{(i)}$ for i=0,1 and f that are needed, namely α_1 and α_2 can be substituted in.

From the definition of transvectant,

$$(f,e_2)^{i}=(-1)^{i}\sum_{j=0}^{i} W_{f,e_2}^{i,j}(\mathbf{Y^j f})((Y^*)^{\mathbf{i-j}}e_2).$$

$$W_{f,e_2}^{i,j} = \binom{i}{j}\frac{(w_f - j)!}{(w_f - i)!}\cdot\frac{(w_{e_2} - i + j)!}{(w_{e_2} - i)!}$$

We have,

,w_f=1 and $w_{e_2}=w_{e_4}$=1 , therefore

$$(f,e_2)^{(0)} = \begin{bmatrix} 0 \\ f \\ 0 \\ 0 \end{bmatrix}$$

To compute $(f,e_2)^{(1)}$,

j=0,i=1, $W_{f,e_2}^{1,0} = \binom{1}{0}\dfrac{(1-0)!}{(1-1)!}\cdot\dfrac{(1-1+0)!}{(1-1)!} = 1$

j=1,i=1, $W_{f,e_2}^{1,1} = \binom{1}{1}\dfrac{(1-1)!}{(1-1)!}\cdot\dfrac{(1-1+1)!}{(1-1)!} = 1$

Thus,

$$(f,e_2)^{(1)} = w_f(Y^*)e_2 - w_f Y fe_2 \quad (9)$$

$$(f,e_2)^{(1)} = - f\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - Yf\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad (10)$$

$$(f,e_2)^{(1)} = \begin{pmatrix}f\\Yf\\0\\0\end{pmatrix} \quad (11)$$

The nonzero constant -1 maybe ignored because we are only concerned with computing basis elements. It can be shown that $w_f f = XYf$ Therefore,

(12)

4.2 *Normal Form for N_{222} System*

The Stanley decomposition of ring of invariants of a system with linear part N_{222} is given by:

$$ker\, X_{222} =$$

$$R[[\alpha_1,\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3}]]\oplus R[[\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3},\beta_{2,3}]]\beta_{2,3}\oplus$$

We have three blocks and therefore the basis elements for N_{222} are e_2,e_4 and e_6. We need to compute the box product of the invariants ring, $ker\, X_{222}$ with $Re_2\oplus Re_4\oplus Re_6$. Thus $ker\, X_{222}=kerX_{222}\boxtimes[Re_2\oplus Re_4\oplus Re_6]$. Let $R=R[[\beta_{1,2},\beta_{1,3}]]$, then

$ker\, X_{222}=[R[[\alpha_1,\alpha_2,\alpha_3]]\oplus R[[\alpha_2,\alpha_3,\beta_{2,3}]]\beta_{2,3}]\boxtimes[Re_2\oplus Re_4\oplus Re_6]$.

Suppressing $\beta_{2,3}$ since it is of weight zero and noting that it shall appear in every square bracket of the second calculation, there are three cases to consider. Computing and simplifying the cases we obtain the normal form of N_{222} as

$$ker\, X_{222} =$$

$$R[[\alpha_1,\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3}]]e_{2r}\oplus R[[\alpha_1,\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3}]](\alpha_1,e_{2r})^{(1)}\oplus$$

$$R[[\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3}]](\alpha_2,e_{2r})^{(i)}\oplus R[[\alpha_3,\beta_{1,2},\beta_{1,3}]](\alpha_3,e_{2r})^{(i)}\oplus$$

$$R[[\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3},\beta_{2,3}]]\beta_{2,3}e_{3r}\oplus R[[\alpha_2,\alpha_3,\beta_{1,2},\beta_{1,3},\beta_{2,3}]]\beta_{2,3}(\alpha_2,e_{2r})^{(i)}$$
$$R[[\alpha_3,\beta_{1,2},\beta_{1,3},\beta_{2,3}]]\beta_{2,3}(\alpha_3,e_{3r})^{(i)} \quad (13)$$

where r=1,2,3 such that $e_{2(1)}=e_2$, $e_{2(2)} = e_4$ and $e_{2(3)}=e_6$.

We compute the transvectants that appear from the cases of the normal form. These are $(f,e_2)^{(i)}$, $(f,e_4)^{(i)}$ and $(f,e_6)^{(i)}$ for i=0,1, where f are the monomials from the Stanley decomposition of the ring of invariants i.e.

$$(f,e_2)^{(0)} = \begin{pmatrix} 0 \\ f \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, (f,e_2)^{(1)} = \begin{pmatrix} XYf \\ Yf \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(f,e_2)^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \\ 0 \\ 0 \end{pmatrix}, (f,e_2)^{(1)} = \begin{pmatrix} 0 \\ 0 \\ XYf \\ Yf \\ 0 \\ 0 \end{pmatrix}$$

$$(f,e_2)^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f \\ 0 \end{pmatrix}, (f,e_2)^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ XYf \\ Yf \end{pmatrix}$$

The Stanley decomposition for the normal form of N_{222} can then summarized as

$$\ker X_{(2)^3} = \bigoplus_{r=1}^{r=3} \left[SD(\ker X_{(2)^3}) \left(\bigoplus_{i=0}^{1} (f, e_{2r})^{(i)} \right) \right]$$

The vector fields $(f,e_{2r})^{(i)}$ are the basis of the normal form of $ker\, X_{(2)^3}$ and f are the standard monomials of the ring of invariants, $ker\, X_{(2)^3}$.

This agrees with the results in [4].

In general from the above examples the normal form of the coupled $N_{22,...,2}$ system is obtained by computing the box product

$$ker\, X_{(2)^n} = ker\, X_{(2)^n} \boxtimes [Re_2 \oplus ... \oplus R_{2n}].$$

The basis of the normal form of $ker\, X_{(2)^n}$ are transvectants of the form: $(f,e_{2r})^{(i)}$ where f is the standard monomials of Stanley decomposition of the ring of invariants, $ker\, X_{(2)^n}$, where i=0,1, and r=1,2,...,n. The Stanley decomposition for the normal form for coupled system $N_{22,...2}$ is given by

$$\ker \mathrm{X}_{(2)^n} = \bigoplus_{r=1}^{r=n} \left[SD(\ker X_{(2)^n}) \left(\bigoplus_{i=0}^{1} (f, e_{2r})^{(i)} \right) \right]$$

where

1. n is the number of Jordan blocks
2. $SD(ker\, X_{(2)^n})$ is the Stanley decomposition of the ring of invariants of $ker\, X_{(2)^n}$.
3. The transvectants $(f,e_{2r})^{(i)}$ are the basis of the normal form of $ker\, X_{(2)^n}$.

References

[1] C. David, L. John, D. O'shea. Ideals, varieties and algorithm, New York, Springer, 1997.

[2] Cushman, J.R. Sanders, N. White, Normal form for the (2;n) nilpotent vector field using invariant theory, Physica D, 1988, 30:399 - 412

[3] David Malonza, Normal forms for coupled Takens-Bogdanov systems, J Nonlinear Math Phy, 2004, 11:376 - 398.

[4] David Malonza, Stanley decomposition for coupled Takens-Bogdanov systems. J Nonlinear Math Phy, 2010, 17: 69-85.

[5] J. Murdock, On the structure of nilpotent normal form modules, J differ Equations, 2002, 180:198-237

[6] J. Murdock, Normal forms and unfolding for local dynamical system, New York, Springer, 2003.

[7] J. Murdock, J.A. Sanders, A New transvectant algorithm for nilpotent normal forms, J Differ Equations, 2007, 238: 234 - 256.

[8] N. Sri Namachchivaya, Normal form for generalized Hopf bifurcation with nonsemisimple 1:1 resonance, Math. Phys, 1994, 45:312 - 335