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Some identities on Strongly Multiplicative Function

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Abstract
A real or complex valued function defined on the set of all positive integers is called an arithmetic function. An arithmetic function f is said to be multiplicative function in one argument if f is not identically zero and f(mn) = f(m) f(n) whenever (m,n) = 1. The objective of this paper is to prove few useful identities using strongly multiplicative functions and using K – Prime integers.
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I. Introduction

An Arithmetic function is a real or complex valued function defined on the set of all positive integers. An arithmetic function f is said to be multiplicative function in one argument if f is not identically zero and f(mn) = f(m) f(n) whenever (m,n) = 1. An arithmetic function is said to be completely multiplicative function if f is not identically zero and f(mn) = f(m)f(n) for all m, n. The function f(m,n) of two variables defined for pairs of positive integers m and n is said to be multiplicative in both the arguments m and n if f(1,1) = 1 and f(m_1m_2,n_1n_2) = f(m_1,m_2)f(m_2,n_2) where (m_1n_1,m_2n_2) = 1.

The Euler Totient function φ (n) is defined to be the number of positive integers not exceeding n which are relatively prime to n . It is given by

(1.1) $\emptyset(n) = \sum_{d/n\,(d,n)=1} 1$

1.2 Definition: *Strongly Multiplicative function*: A multiplicative arithmetic function f is said to be strongly multiplicative function if for every prime P, we have $f(p) = f(p^2) = f(p^3) = \cdots \ldots \ldots \ldots \ldots$

If a and b are integers, not both zero, and k is any integer greater than 1, then $(a,b)_k$ denotes the largest common divisor of a and b which is also a k^{th} power. This will be referred to as k^{th} power greatest common divisor of a and b.

1.3 Definition: If $(a,b)_k = 1$, then a is said to be ***relatively K – Prime*** to b.

1.4 Ecford Cohen [3] introduced a function $\emptyset_k(n)$ which denotes the number of non negative integers less than N^k which are relatively K-Prime to N^k.

(1.5) $\sum_{d/n} \emptyset_k(N/d) = N^k$

1.6 Definition: The Mobius function μ (n) is defined by

$$\mu(n) = \{1 \quad if\ n = 1\ (-1)^k\ if\ n = p_1p_2\ldots p_k,\ 0 \quad otherwise$$

where p_i's are distinct primes.

By the Mobius inversion formula, we get that

(1.7) $\emptyset_k(N) = \sum_{d/N} d^k\,\mu(N/d)$

II. Preliminaries

We will derive few preliminary lemmas which will be used to derive the final result.

2.1 lemma: $\phi_k(r)$ is multiplicative function.

Proof: In view of (1.5), we get

$$\phi_k(r) = \sum_{(r)} d^k \mu\left(\frac{r}{d}\right)$$

In order to prove multiplicative, we have to show that

$\phi_k(r_1 r_2) = \phi_k(r_1)\,\phi_k(r_2)$ whenever $(r_1 r_2) = 1$

Suppose that r_1, r_2 are positive integers such that $(r_1 r_2) = 1$. Then

(2.2) $\phi_k(r_1 r_2) = \sum_{(r_1 r_2)} d^k \mu\left(\frac{r_1 r_2}{d}\right)$

Every divisor d of $r_1 r_2$ can be uniquely written as d = $d_1 d_2$ where $\frac{d_1}{r_1}$ *and* $\frac{d_2}{r_2}$

Then (2.2) can be written as

$$\phi_k(r_1 r_2) = \sum_{(r_1 r_2)} (d_1 d_2)^k \mu\left(\frac{r_1 r_2}{d_1 d_2}\right)$$

$$= \sum_{(r_1)} \sum_{(r_2)} d_1^k d_2^k\, \mu\left(\frac{r_1}{d_1}\right) \mu\left(\frac{r_2}{d_2}\right)$$

$$= \left(\sum_{(r_1)} d_1^k \mu\left(\frac{r_1}{d_1}\right)\right)\left(\sum_{(r_2)} d_2^k\, \mu\left(\frac{r_2}{d_2}\right)\right)$$

$$= \phi_k(r_1)\,\phi_k(r_2)$$

Proving the lemma.

2.3 Lemma: Let $f(r) = \frac{r^k}{\phi_k(r)}$. Then $f(r)$ is strongly multiplicative function.

Proof: First we see that f is a multiplicative function.

That is, $f(r_1 r_2) = f(r_1) f(r_2)$

Consider $f(r_1 r_2) = \frac{(r_1 r_2)^k}{\emptyset_k(r_1 r_2)}$

$= \frac{r_1^k\, r_2^k}{(\phi_k(r_1)\,\phi_k(r_2))}$ in view of lemma 2.1

$= \left(\frac{r_1^k}{\phi_k(r_1)}\right)\left(\frac{r_2^k}{\phi_k(r_2)}\right)$

$= f(r_1) f(r_2)$

Thus $f(r)$ is multiplicative function.

Also, we have for every prime p,

$$f(p) = \frac{P^k}{\phi_k(P)}$$

$$= \frac{P^k}{\sum_{\frac{d}{p}} d^k \mu\left(\frac{p}{d}\right)}$$

$$= \frac{P^k}{1^k \mu(P) + P^k \mu(1)}$$

$$= \frac{P^k}{P^k - 1}$$

and

$$f(P^2) = \frac{P^{2k}}{\phi_k(P^2)}$$

$$= \frac{P^{2k}}{\sum_{(p^2)} d^k \mu\left(\frac{p^2}{d}\right)}$$

$$= \frac{P^{2k}}{1^k \mu(P^2) + P^k \mu(P) + P^{2k} \mu(1)}$$

$$= \frac{P^{2k}}{-P^k + P^{2k}}$$

$$= \frac{P^k}{P^k - 1}$$

Similarly it can be shown that

$$f(P^3) = f(P^4) = \cdots = \frac{P^k}{P^k - 1}$$

Thus proving that $f(r)$ is strongly multiplication function.

III. Main Result

3.1 Theorem: Let $g(r) = \frac{1}{\phi_k(r)}$. Then $\frac{P^k}{\phi_k(P)} - 1 = \frac{1}{\phi_k(p)}$

Proof: Consider $g(p) = \frac{1}{\phi_k(p)}$

$$= \frac{1}{\sum_{\frac{d}{p}} d^k \mu\left(\frac{P}{d}\right)} \text{, from (1.5)}$$

$$= \frac{1}{1^k \mu(P) + P^k \mu(1)}$$

$$= \frac{1}{P^k - 1}$$

and consider $\frac{P^k}{\phi_k(P)} - 1$

$$= \frac{P^k}{P^k - 1} - 1$$

$$= \frac{1}{P^k - 1}$$

Thus proving the result.

3.2 Theorem: $\sum_{(r)} \frac{d^k}{\phi_k(d)} \mu\left(\frac{r}{d}\right) = \mu(r)\, \mu\left(\frac{r}{t}\right) \frac{1}{\phi_k\left(\frac{r}{t}\right)}$

Proof: In view of lemma 2.3, we have $f(r) = \frac{r^k}{\phi_k(r)}$ which gives $f(d) = \frac{d^k}{\phi_k(d)}$

and ([5], Theorem 3.1) gives for $n \geq 1, m \geq 1$, we have

(3.3) $$\sum_{\substack{d|n \\ (d,m)=1}} f(d)\mu\left(\frac{n}{d}\right) = \mu(n)\mu\left(\frac{n}{t}\right) h\left(\frac{n}{t}\right), \text{ where } t = (n, m)$$

using (3.3), $\sum_{(r)} \frac{d^k}{\phi_k(d)} \mu\left(\frac{r}{d}\right) = \mu(r)\mu\left(\frac{r}{t}\right) h\left(\frac{n}{t}\right)$

$$= \mu(r)\mu\left(\frac{r}{t}\right) \frac{1}{\phi_k\left(\frac{r}{t}\right)}, \text{ from theorem 3.1.}$$

Hence the result.

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