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Line Graph of Colored Graph

Shigehalli V.S[1]. and Kenchappa Betageri[2]
[1]Professor, Dept. of Mathematics, Rani Channamma University, Belagavi, Karnataka, India,
Email: shigehallivs@yahoo.co.in
[2] Research Scholar, P.G. Studies in Dept. of Mathematics, Karnatak University Dharwad,
Karnataka, India. Email: kmath.bet@gmail.com

ABSTRACT: In this article we introduce new class of graphs called line graph of colored graphs and is denoted by , involves two important graph parameters viz., edge set and vertex coloring of graph. The line graph of a colored graph is the simple graph whose vertices are the edges of with two vertices in are adjacent, whenever the corresponding edges in share at least one common colored vertex in . We discuss some fundamental properties and characterizations of .
Key words: Colored graphs, Line graph of colored graphs.

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1.0 Introduction

Graph theory is young and rapidly maturing subject. It is always changing considerably at every quarter of the century. We have found that there is a clear need for a text only to the well established results but too many of the newer developments as well. Basic concepts of graph theory are extraordinary simple and can be used to express problems from many different subjects.

A graph is a pair of sets , where is finite non-empty set of elements called vertices, and E is a set of unordered pairs of distinct vertices called edges. The sets and are vertex set and edge set of and are often denoted by and , respectively.

The concept of line graph of a given graph is so natural that it has been independently discovered by many researchers in [2,3,4,5,6,7].

Definition 1. (Line Graph/Edge Graph): Let be a simple graph with be the vertex set and be edge set of . The line graph of graph is a simple graph with the vertices in which and are adjacent whenever, the edges and share a common vertex in . The is called the line graph or edge graph of graph .

Various properties and characterization of line graphs are developed in the literature, reader can refer [8-11].

The notion of graph coloring originates from coloring of countries of a map where each face is virtually colored. This was extended to coloring the faces of a graph embedded in a plane. By planar duality it become coloring the vertices of and in this form it generalize to any graphs. The concept of graph coloring was introduced in 1852 with a four color problem and it becomes a famous problem in graph theory. Later graph coloring has been studied as an algorithm problem since 1970.

A proper coloring of is an assignment of colors to each vertex so that adjacent vertices receive different colors, If the vertices have been colored with colors then the graph is said to be coloring. The chromatic number of is the minimum number of colors required to color vertices of the graph, such that no two adjacent vertices have the same color. It is denoted by .

Note that, an efficient (polynomial time) algorithm is not yet found for proper coloring of arbitrary graphs.

We consider finite, undirected colored graphs throughout the paper. We have large enough literature on two graph parameters viz., edge set and vertex coloring , of graph. For more details on these two parameters see [11,12, 13].

Note 1: The graphs obtained by proper coloring is called colored graphs (denoted by) and arbitrarily coloring is not unique.

Definition 2. (Hamiltonian Graph): A Hamiltonian cycle of a graph is cycle of length , i.e. the cycle goes through all vertices once. A graph is called Hamiltonian if it consists a Hamiltonian cycle.

This article is organized as follows: Section 1.0 deals with brief introduction of line graph and coloring of graph. Section 1.1 deals with definition of line graph of a colored graph. i.e., and some fundamental properties of .

1.1 Main Results

In this section we define the concept called line graph of a colored graph and discuss some basic properties.

The line graph of a colored graph is denoted by and is the simple graph whose vertices are the edges of with two colored vertices in are adjacent whenever the corresponding edges in share at least one common colored vertex.

Note that, in general the optimal coloring with colors is not unique, so for different optimal coloring of graph, we get different line graphs of . Trivial fact is, for any colored graph with , then is a complete graph of order, where is the number of edges in . It suffices that will construct a huge class of isomorphic graphs from old non- isomorphic colored graphs, exclusively for with .

The concept of color line graph gives some quite surprising results and hopefully, that can be applied to some specific real world problems, like, analysis of storage problems, design theory etc., and also construction of new graphs from old graph, finding isomorphism between newly constructed graphs is of good academic interest, so we have define and discuss some basic properties of . The subgraphs and forbidden subgraphs of , isomorphism between newly constructed graphs are discussed in our forth coming two articles.

The article is designed as follows.

Definition 3. (Line Graph of a Colored Graph): Let be a simple colored graph with colors and be the vertex set, be edge set of . The line graph of a colored graph is a simple graph with the vertices in which and are adjacent whenever, the edges and share at least one common colored vertex in . The is called the line graph or edge graph of colored graph .

In other words, the line graph of a colored graph is the simple graph whose colored vertices are the edges of with two colored vertices in are adjacent, whenever the corresponding edges in share at least one common colored vertex.

Note that is always simple and isolated vertices of do not have any bearing on , so we assume that has no isolated vertices.

Properties of Line Graph of a Colored Graph.

In this section we discuss basic properties of line graph of colored graphs. Throughout the discussion for any edges such that means no edges share at least one common colored vertex in and share at least one common colored vertex in .

Observations

1. Line graph is spanning graph of line graph of a colored graph . i.e., .
2. For any proper colored graph with for some , then is connected graph.
3. The maximum edges share a common colored vertex in , givesrise to a clique (complete subgraph) of line graph of .
4. If be a complete colored graph of order with colors, then   has clique of order .

Proof of observation 4. Let with and for all, implies, except disjoint edges of . Hence by definition of the corresponding vertices and are adjacent in except edges such that . Therefore has clique of order .

The following Theorem 1 is trivial.

Theorem 1. if and only if with .

Proof. Suppose be a colored graph with , implies ,for all . Thus by definition of the corresponding every pair of vertices are adjacent in for all and hence .

Conversely, Let , it follows that every pair of vertices in are joined by an edge, implies ,for all , whence .

On the contrary, suppose , then there exist at least one pair of disjoint edge , such that , implies the corresponding edges and are not adjacent in . This contradicts to the fact that . Therefore the colored graph must be .

Corollary 1. If if, and only if

By Theorem 1, it is evident that, if is colored graph with , then is Hamiltonian.

Theorem 2. If is line graph of colored graph , then, the number of edges sharing a common colored vertices in is and degree of in is , where is the number of disjoint edges in those share at least one common colored vertex in .

Proof. If is an edge of simple colored graph joining two colored vertices and , then degree of in is same as the number of edges sharing at least one common colored vertex in . Therefore this number is precisely, , where is the number of disjoint edges of those share at least one common colored vertex in .

Hence .

Now here two cases arises, case (i) and (ii) .

case (i). If then the line graph of a colored graph coincides with the ordinary line graph of underlying graph . Therefore and

Implies ,

where is the degree sequence of the and is the number of edges in . Therefore by Euler's theorem (i.e., sum of the degree of the vertices of a graph is equal to twice the number of edges), it follows that

.

We have the following are some special class of colored graphs, whose

(i). complete bipartite colored graphs and complete multipartite colored graphs.

If The similar argument as discussed above follows with addition of .

(1)

Now have again two cases,

Case a). Equation (1) holds if for any two disjoint edges such that then . Otherwise, equation (1) reduces to

(2)

Case b). Let any two edges do not share at least one common colored vertex in . i.e., otherwise .

(3)

(Since belongs to star at and , and with addition of), it implies

Implies

Again by Euler's Theorem and with addition of in , it follows that

. (4)

This completes the proof of the theorem.

The following Theorem 3, gives us the bound for the number of vertices and edges in in first Zagreb index and .

Theorem 3. For any colored graph , with , for all . Then

i.
ii. E (5)

Where and

Proof: By the definition of , (i) follows.

To prove (ii). Let be a graph with by observation1, is subgraph of . Hence .,

. i.e., . (6)

Since every edge is non incident with exactly edges. Therefore by (6), we have

E

This completes the proof of (ii).

Further if , then by Theorem 1, equality holds in (i).

Corollary 1. Let graph with and clique no , then

E (7)

Proof. Proof follows from observation 3 and (ii) of Theorem 3, above.

Conclusion: We assume and have strong belief that will be more helpful in the analysis of storage problems, design theory etc., and in future we work on its applications.

References

1. Ore, Theory of Graphs. Amer. Math. Soc. Colloque. Publi, 38, Providence 1962.
2. Sabidussi G., Graph derivatives, Math. Z. 76 (1961).
3. L. Beineke., Derived graph and diagraphs, Beitrage zur Graphen theorie, [H. Sachs, H. Voss and H. Walther eds.] Teubaer Leipzig, 1968. p.17-33.
4. Seshu S, and M.Reed., Linear Graphs and Electrical Networks, Addison Wesley Reading. (1963).
5. Kasteleyn P.W., Graph Theory and Crystal Physics, Chapter 2, in Graph theory and Theoretical Physics (F. Harary. ed), Academic Press, London. 1967, p. 44-110.
6. Menon V., On repeated interchange graphs. Amer. Math Montly, 13(1966), 986- 989.
7. Krausz J., Demonstration nouvelle d'une th´eor`eme de Whitney sur les reseaux Mat. Fiz. Lapok 50(1943), 75–89.
8. F. Harare, Graph Theory. Addison Wisely Publication.
9. D. Cvetkovic, Peter Rowilson., S. Simi., Spectral Generalization of line graphs. Cambridge University Press, London Mathematical Society Lecture Note Series 314.
10. Beineke L. W., Characterization of derived graphs, J. Combin. Theory 9,(1970), 129–135.
11. F. Harary, "Graph theory" Narosa Publishing House New Delhi 2001.
12. J.A. Bondy and U.S.R. Murty. Graph Theory, Graduate text in Mathematics, Springer 2004.
13. Gary Chartrand and Ping Zhang, Chromatic Graph Theory, Discrete Mathematics and Its Applications.